5. Differentials of First and Higher Orders

5.1 Increment of the argument and increment of a function.

If x and x_1 are the values of the argument *x* and $y = f(x)$ and $y_1 = f(x_1)$ are the corresponding values of the function *y = f*(*x*), then

$$\Delta x = x_1 - x$$

is called the increment of the argument *x* in the interval (x, x_1) and

$$\Delta y = y_1 - y \quad or \quad \Delta y = f(x_1) - f(x) = f(x + \Delta x) - f(x) \qquad (1)$$

is called the increment of the function *y* in the same interval (x, x_1) (Fig, 1, where $\Delta x = MA$ and $\Delta y = AN$).

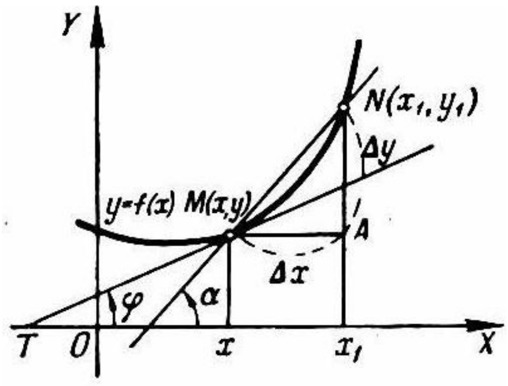

Fig. 1

The ratio $\frac{\Delta y}{\Delta x} = \tan\alpha$ is the slope of the secant *MN* of the graph of the function *y = f(x)* and is called the mean rate of change of the function *y* over the interval $(x, x + \Delta x)$.

Example 1. Calculate for the function $y = x^2 - 5x + 6$ values Δx and Δy corresponding to a change in the argument: a) from $x = 1$ to $x = 1.1$, b) from $x = 3$ to $x = 2$.

Solution: We have

$a)\ \Delta x = 1{,}1 - 1 = 0{,}1;\quad \Delta y = (1{,}1^2 - 5\cdot 1{,}1 + 6) - (1^2 - 5\cdot 1 + 6) = -0{,}29;$

$b)\ \Delta x = 2 - 3 = -1;\quad \Delta y = (2^2 - 5\cdot 2 + 6) - (3^2 - 5\cdot 3 + 6) = 0.$

Example 2. Find for the hyperbola y = 1/x the slope of the secant passing through the points *M*(3, 1/3) and *N* (10, 1/10).

Solution: Here,

$$\Delta x = 10 - 3 = 7, \Delta y = \frac{1}{10} - \frac{1}{3} = -\frac{7}{30}, whence\ k = \frac{\Delta y}{\Delta x} = -\frac{1}{30}.$$

5.2 First-order differential

The differential (first-order) of a function is the principal part of its increment, which part is linear relative to the increment $\Delta x = dx$ of the independent variable *x*. The differential of a function is equal to the product of its derivative and the differential of the independent variable $dy = y'dx$, whence $y' = \frac{dy}{dx}$.

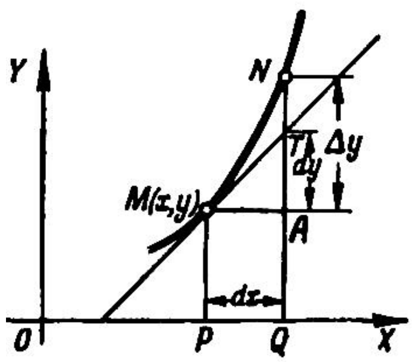

Fig. 19

If MN is an arc of the graph of the function $y = f(x)$ (Fig. 19), MT is the tangent at $M(x, y)$ and $PQ = \Delta x = dx$ then the increment in the ordinate of the tangent $AT = dy$ and the segment $AN = \Delta y$.

Example. Calculate the increment of the function $y = 3(2 - x)^2$ for $x_1 = 3$ and $x_1 = 3.001$

Solution: $\Delta y = 3(2 - 3.001)^2 - 3(2 - 3)^2 = 0{,}006003$

Example. Calculate the increment of the function $y = \sqrt{x^2 - x}$ for $x_1 = 2$ and $\Delta x = 0.03$

Solution:

Example. Find the increment and the differential of the function y = 3x² - *x*.

Solution:

$\Delta y = 3(x+\Delta x)^2 - (x+\Delta x) - 3x^2 + x$ or $\Delta y = (6x-1)\Delta x + 3(\Delta x)^2$

$\Delta y = 3(x+\Delta x)^2 - (x+\Delta x) - 3x^2 + x$ or $\Delta y = (6x-1)\Delta x + 3(\Delta x)^2$,

whence

$dy = (6x-1)dx = (6x-1)\Delta x$.

Example. Calculate Δy and *dy* of the function *y* = 3x² - *x* for x= 1 and $\Delta x = 0{,}01$.

Solution: $\Delta y = (6x-1)\Delta x + 3(\Delta x)^2 = 5\cdot 0.01 + 3(0.01)^2 = 0.0503$,

$$dy = (6x-1)\Delta x = 5\cdot 0{,}01 = 0{,}0500.$$

5.3. Applying differentials to approximate calculations

If the increment Δx of the argument *x* is small in absolute value, then the differential *dy* and the increment ▯y of the function *y* = *f*(*x*) are approximately the same:

$$\Delta y \approx dy, \text{ i.e., } f(x+\Delta x) - f(x) \approx f'(x)\Delta x, \text{ whence } f(x+\Delta x) \approx f(x) + f'(x)dx.$$

Example. By (approximately) how much does the side of a square change if its area increases from 9 m² to 9.1 m² ?

Solution: If *x* is the area of the square and *y* is its side, then $y = \sqrt{x}$. We know that x = 9 and $\Delta x = 0{,}1$. The increment Δy of the side of the square may be calculated approximately as follows:

$$\Delta y \approx dy = y'\Delta x = \frac{1}{2\sqrt{9}}\cdot 0{,}1 = 0{,}016\, m.$$

5.4 Principal properties of differentials:

- ✓ $dc = 0$, where *c*=const
- ✓ $dx = \Delta x$, where x is an independent variable
- ✓ $d(cu) = cdu.$
- ✓ $d(u \pm v) = du \pm dv$
- ✓ $d(uv) = udv + vdu$
- ✓ $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$
- ✓ $df(u) = f'(u)du.$

5.5 Higher order differentials

A second-order differential is the differential of a first-order differential:

$$d^2y = d(dy).$$

We define similarly the differentials of the third and higher orders. If $y = f(x)$ and x is the independent variable, then

$$d^2y = y''(dx)^2,$$
$$d^3y = y'''(dx)^3,$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$d^ny = y^{(n)}(dx)^n.$$

However, if $y = f(u)$, where $u = u(x)$, then

$$d^2y = y''(du)^2 + y'\,d^2u,$$
$$d^3y = y'''(du)^3 + 3y''\,du \cdot d^2u + y'\,d^3u, \text{ etc.}$$

(Here the primes denote derivatives with respect to *u*).

Exercises 712-715, 722-734, 747-755

712. Find the increment Δy and the differential *dy* of the function $y = 5x + x^2$ for $x = 2$ and $\Delta x = 0.001$.

713. Without calculating the derivative, find $d(1 - x^2)$ for x=1 and $\Delta x = -\frac{1}{3}$.

714. The area of a square S with side *x* is given by $S = x^2$. Find the increment and the differential of this function and explain the geometric significance of the latter.

715. Give a geometric interpretation of the increment and differential of the functions: a) the area of a circle, $S = \pi x^2$, b) the volume of a cube, $V = x^3$.

In the following problems, find the differentials of the given functions for arbitrary values of the argument and its increment:

722. $y = \frac{1}{x^m}$.
723. $y = \frac{x}{1-x}$.
724. $y = \arcsin \frac{x}{a}$.
725. $y = \arctan \frac{x}{a}$.
726. $y = e^{-x^2}$.
727. $y = x \ln x - x$.
728. $y = \ln \frac{1-x}{1+x}$.
729. $r = \cot \varphi + \operatorname{cosec} \varphi$.
730. $s = \arctan e^t$.

Answers:

722. $\frac{-m\,dx}{x^{m+1}}$. 723. $\frac{dx}{(1-x)^2}$. 724. $\frac{dx}{\sqrt{a^2 - x^2}}$.
725. $\frac{a\,dx}{x^2 + a^2}$. 726. $-2xe^{-x^2}\,dx$. 727. $\ln x\,dx$. 728. $\frac{-2dx}{1-x^2}$. 729. $-\frac{1+\cos\varphi}{\sin^2\varphi}\,d\varphi$.
730. $-\frac{e^t dt}{1+e^{2t}}$. 732. $-\frac{10x+8y}{7x+5y}\,dx$. 733. $\frac{-ye^{-\frac{x}{y}}\,dx}{y^2 - xe^{-\frac{x}{y}}} = \frac{y}{x-y}\,dx$. 734 $\frac{x+y}{x-y}\,dx$.

731. Find *dy* if $x^2 + 2xy - y^2 = a^2$.

Solution: Taking advantage of the invariance of the form of a differential, we obtain 2xdx + 2(ydx + xdy) = 2ydy = 0, whence

$$dy = -\frac{x+y}{x-y}\,dx.$$

In the following examples find the differentials of functions defined implicitly.

732. $(x+y)^2 \cdot (2x+y)^3 = 1$. **733.** $y = e^{-\frac{x}{y}}$. **734.** $ln\sqrt{x^2+y^2} = arctan\,\frac{y}{x}$.

747. Compute d^2y, if y = cos 5*x.*

Solution: $d^2y = y''(dx^2) = -25cos\,5x(dx)^2$.

748. $u=\sqrt{1-x^2}$, find d^2u.
749. $y=\arccos x$, find d^2y.
750. $y=\sin x \ln x$, find d^2y.
751. $z=\frac{\ln x}{x}$, find d^2z.
752. $z=x^2e^{-x}$, find d^3z.
753. $z=\frac{x^4}{2-x}$, find d^4z.
754. $u=3\sin(2x+5)$, find d^nu.
755. $y=e^{x\cos\alpha}\sin(x\sin\alpha)$, find d^ny.

Answers 712-715, 722-734, 747-755

712. $\Delta y=0.009001$; $dy=0.009$. 713. $d(1-x^3)=1$ when $x=1$ and $\Delta x=-\frac{1}{3}$.
714. $dS=2x\,\Delta x$, $\Delta S=2x\,\Delta x+(\Delta x)^2$.

722. $\frac{-mdx}{x^{m+1}}$. 723. $\frac{dx}{(1-x)^2}$. 724. $\frac{dx}{\sqrt{a^2-x^2}}$.

725. $\frac{a\,dx}{x^2+a^2}$. 726. $-2xe^{-x^2}\,dx$. 727. $\ln x\,dx$. 728. $\frac{-2dx}{1-x^2}$. 729. $-\frac{1+\cos\varphi}{\sin^2\varphi}\,d\varphi$.

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748. $\frac{-(dx)^2}{(1-x^2)^{3/2}}$. 749. $\frac{-x\,(dx)^2}{(1-x^2)^{3/2}}$. 750. $\left(-\sin x\ln x+\frac{2\cos x}{x}-\frac{\sin x}{x^2}\right)(dx)^2$.

751. $\frac{2\ln x-3}{x^3}(dx)^2$. 752. $-e^{-x}\times(x^2-6x+6)\,(dx)^3$. 753. $\frac{384\,(dx)^4}{(2-x)^5}$.

754. $3\cdot 2^n\sin\left(2x+5+\frac{n\pi}{2}\right)(dx)^n$.