

6.1 Equations of the tangent and the normal

It follows from the geometric significance of the derivative that the equation of the **tangent** to a curve $y = f(x)$ or $F(x, y) = 0$ at a point $M(x_0, y_0)$ will be

$$y - y_0 = y_0'(x - x_0),$$

where y_0' is the value of the derivative y' at the point $M(x_0, y_0)$. The straight line passing through the point where the tangent touches the curve, perpendicularly to the tangent, is called the **normal** to the curve. The normal has the equation

$$y - y_0 + y_0'(x - x_0) = 0$$

Example 1. Write the equation of the tangent and the normal to the parabola $y = \sqrt{x}$ at the point with abscissa $x = 4$.

Solution: We have $y' = \frac{1}{2\sqrt{x}}$, whence the slope of the tangent is $k = |y'|_{x=4} = 1/4$. Since the point of the tangent has the co-ordinates $x = 4, y = 2$, the equation of the tangent is $y - 2 = 1/4(x - 4)$ or

$$x - 4y + 4 = 0$$

Since the slope of the normal must be perpendicular, $k_1 = -4$, whence the equation of the normal is

$$y - 2 = -4(x - 4) \quad \text{or} \quad 4x + y - 18 = 0.$$

Exercises

1. Write the equations of the tangent and the normal to the curve $y = x^3 + 2x^2 - 4x - 3$ at the point $(-2, 5)$. 2. Find the equations of the tangent and the normal to the curve $y = \sqrt[3]{x-1}$ at the point $(1, 0)$. 3. Form the equations of the tangent and the normal to the curves at the given points: a) $y = \tan 2x$ at the origin; b) $y = \arcsin[(x-1)/2]$ at the intersection with the x -axis; c) $y = \arcsin 3x$ at the intersection with the y -axis; d) $y = \ln x$ at the intersection with the x -axis; e) $y = e^{1-x^2}$ at the intersection with the straight line $y = 1$. 4. Write down the equations of the tangent and the normal to the curve $x = \frac{1+t}{t^3}, y = \frac{3}{2t^2} + \frac{1}{2t}$ at the point $(2, 2)$. 5. Find the equations of the tangent to the curve $x = t \cos t, y = t \sin t$ at the origin and the point $t = \pi/4$. 6. Find the equations of the tangent and the normal to the curve $x^3 + y^2 + 2x - 6 = 0$ at the point with ordinate $y = 3$. 7. Find the equation of the tangent to the curve $x^5 + y^5 - 2xy = 0$ at the point $(1, 1)$. 8. Find the equations of the tangents and normals to the curve $y = (x-1)(x-2)(x-3)$ at its intersection with the x -axis. 9. Find the equations of the tangent and the normal to the curve $y^4 = 4x^4 + 6xy$ at the point $(1, 2)$. 10. Find the points at which the tangents to the curve $y = 3x^4 + 4x^3 - 12x^2 + 20$ are parallel to the x -axis. 626. At what point are the tangent to the parabola $y = x^2 - 7x + 3$ and the straight line $5x + y - 3 = 0$ parallel? 627. Find the equation of the parabola $y = x^2 + bx + c$ which is tangent to the straight line $x = y$ at the point $(1, 1)$. 628. Determine the slope of the tangent to the curve $x^3 + y^3 - xy - 7 = 0$ at the point $(1, 2)$. 629. At what point of the curve $y^2 = 2x^3$ is the tangent perpendicular to the straight line $4x - 3y + 2 = 0$?

Answers

1. $y - 5 = 0; x + 2 = 0$. 2. $x - 1 = 0; y = 0$. 3. a) $y = 2x; y = -0.5x$; b) $x - 2y - 1 = 0; 2x + y - 2 = 0$; c) $6x + 2y - \pi; 2x - 6y + 3\pi = 0$; d) $y = x - 1; y = 1 - x$; e) $2x + y - 3 = 0; x - 2y + 1 = 0$ for the point $(1, 1)$; $2x - y + 3 = 0; x + 2y - 1 = 0$ for the point $(-1, 1)$. 4. $7x - 10y + 6 = 0; 10x + 7y - 34 = 0$. 5. $y = 0; (\pi + 4)x + (\pi - 4)y - \frac{\pi^2\sqrt{2}}{4} = 0$. 6. $5x + 6y - 13 = 0, 6x - 5y + 21 = 0$. 7. $x + y - 2 = 0$. 8. At the point $(1, 0)$: $y = 2x - 2; y = \frac{1-x}{2}$; at the point $(2, 0)$: $y = -x + 2; y = x - 2$; at the point $(3, 0)$: $y = 2x - 6; y = \frac{3-x}{2}$. 9. $14x - 13y + 12 = 0; 13x + 14y - 41 = 0$. 10. $(0, 20); (1, 15); (-2, -12)$. 11. $(1, -3)$. 12. $y = x^2 - x + 1$. 13. $k = -\frac{1}{11}$. 14. $(\frac{1}{8}, -\frac{1}{16})$.