1.2. Derivative of composite function

1.2.1. Chain rule

The chain rule tells us how to differentiate composite functions

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

If $y = f(u)$ and $u = u(x)$, i.e., $y = f[u(x)]$, where the functions y and u have derivatives, then

$$y'_x = y'_u \cdot u'_x$$

This rule extends to a series of any finite number of differentiable functions.

Common mistake: Wrong identification of the inner and outer function

Even when a student recognized that a function is composite, they might get the inner and the outer functions wrong. This will surely end in a wrong derivative.

In the composite function $y = (5 - 6x)^5$. Students are often confused by this sort of function

$$\underbrace{(\overbrace{5 - 6x}^{\text{inner}})^5}_{\text{outer}}$$

$$f(g(x))' = (\text{outer function})' \cdot (\text{inner function})'$$

Example 1. Find the derivative of the function $y = (3x^2 - 5)^3$.

Solution: The first step is always to recognise that we are dealing with a composite function and then to split up the composite function into its components. In this case the outside function is $(\cdot)^3$ which has derivative $3(\cdot)^2$, and the inside function is $3x^2 - 5$ which has derivative $6x$, and so by the composite function rule,

$$y' = 3(3x^2 - 5)^2 \cdot 6x = 18x(3x^2 - 5)^2.$$

Alternatively we could first let $u = 3x^2 - 5$ and then $y = u^3$. So

$$y' = 3u^2 \cdot 6x = 18x(3x^2 - 5)^2.$$

Example 2. Find the derivative of the function $y = (x^2 - 2x + 3)^5$.

Solution: Setting $y = u^5$, where $u = (x^2 - 2x + 3)$. So

$$y' = (u^5)'_u \cdot (x^2 - 2x + 3)'_x = 5u^4 \cdot (2x - 2) = 10(x - 1)(x^2 - 2x + 3)^4$$

Example 3. Find the derivative of the function $y = \sqrt{x^2 + 1}$

Solution: The outside function is $\sqrt{\cdot} = (\cdot)^{\frac{1}{2}}$ which has derivative $\frac{1}{2}(\cdot)^{-\frac{1}{2}}$, and the inside function is $x^2 + 1$, so that $y' = \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot 2x$

Alternatively, if $u = x^2 + 1$, we have $y = \sqrt{u} = u^{\frac{1}{2}}$. So

$$y' = \frac{1}{2}u^{-\frac{1}{2}} \cdot 2x = \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot 2x$$

Example 4. Find the derivative of the function $y = sin^3 4x$

Solution: Setting $y = u^3, u = \sin v, v = 4x$,we find

$$y' = 3u^2 \cdot \cos v \cdot 4 = 12\sin^2 4x \cdot \cos 4x.$$

Exercises. Find the derivatives of the following functions.

1) $y=(5-6x)^{12}$ 2) $y = (10x^2 - 4x + 1)^5$ 3) $y=\sqrt{11-12x^3+x}$ 4) $y=\sqrt[3]{3x^5-7x+\cos\pi}$

5) $y = 7\ln(1 + x^2)$ 6) $y=\ln\frac{1}{x}$ 7) $y=\ln\sqrt{1-x}$ 8) $y = \ln(\sin x)$

9) $y = \ln^2(4x^2 - 16)$ 10) $y=\sqrt{\ln x}$ 11) $y = \log(\tan 9x)$ 12) $y = \log_2(5x^3 - 1)$

13) $y = e^{-x}$ 14) $y = 8e^{4+3x}$ 15) $y=e^{5x^2-x}$ 16) $y=\sqrt{1-e^{2x}}$

17) $y = e^{2-\sin x}$ 18) $y = 9e^{\cot x}$ 19) $y=\frac{1}{e^{3x}}$ 20) $y = \frac{7}{(6-e^x)^2}$

21) $y = 4^{-x}$ 22) $y=7^{2x^2+3}$ 23) $y=10^{\ln^2 x}$ 24) $y = \sin(7x + 3)$

25) $y = \cos(1 - 6x)$ 26) $y = \tan(4x^2 + 3)$ 27) $y=\cot\sqrt{x}$ 28) $y=\sin^2\frac{1}{x}$

29) $y=\cos^3\sqrt[3]{1-x}$ 30) $y=\tan^4(1-e^x)$ 31) $y=\cot^5(\ln x)$ 32) $y=\frac{1}{\sin x}$

33) $y=\frac{2}{\cos^2 3x}$ 34) $y=\sqrt{\frac{1-x}{1+x}}$ 35) $S=\sqrt{2-t}\,(11t^2+12)^{13}$ 36) $S=\frac{t}{\sqrt{16-t^2}}$

37) $S=\sin^3 t^2$ 38) $S=-2\sqrt{\cos 3x}$ 39) $S=\sin e^{2\varphi}$ 40) $S=34-e^{\sin\varphi}\cdot\cos\varphi$

41) $S=\ln\frac{e^t}{1+e^t}$ 42) $y = \arcsin 2x$ 43) $S=2\arccos\frac{t}{16}$ 44) $S=\arctan\sqrt{15t}$

45) $S=14\,arc\cot\frac{13}{t}$ 46) $S=\sqrt{\arcsin x^2}$

Answers:

1) $-72(5-6x)^{11}$ 2) $5(10x^2-4x+1)^4\cdot(20x-4)$ 3) $\frac{-36x^2+1}{2\sqrt{11-12x^3+x}}$ 4) $\frac{15x^4-7}{3\sqrt[3]{(3x^5-7x+\cos\pi)^2}}$

5) $\frac{14x}{1+x^2}$ 6) $x\cdot\left(-\frac{1}{x^2}\right)=-\frac{1}{x}$ 7) $\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{1-x}}\cdot(-1)=\frac{-1}{2(1-x)}$ 8) $\frac{1}{\sin x}\cdot\cos x=\cot x$

9) $2\ln(4x^2-16)\cdot\frac{1}{4x^2-16}\cdot 8x=\frac{4x\ln(4x^2-16)}{x^2-4}$ 10) $\frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x}=\frac{1}{2x\sqrt{\ln x}}$ 11) $\frac{1}{\tan 9x\cdot\ln 10}\cdot\frac{1}{\cos^2 9x}\cdot 9$

12) $\frac{15x^2}{(5x^3-1)\ln 2}$ 13) $e^{-x}\cdot(-1) = -e^{-x}$ 14) $8e^{4+3x}\cdot 3 = 24e^{4+3x}$ 15) $e^{5x^2-x}\cdot(10x-1)$

16) $\frac{-e^{2x}}{\sqrt{1-e^{2x}}}$ 17) $e^{2-\sin x}(-\cos x)$ 18) $9e^{\cot x}\cdot\left(-\frac{1}{\sin^2 x}\right)$ 19) $e^{-3x}(-3) = -3e^{-3x}$ 20) $\frac{14e^x}{(6-e^x)^3}$

21) $4^{-x}\cdot\ln 4\cdot(-1)$ 22) $7^{2x^2+3}\ln 7\cdot 4x$ 23) $10^{\ln^2 x}\ln 10\cdot 2\ln x\cdot\frac{1}{x}$ 24) $7\cos(7x+3)$ 25) $6\sin(1-6x)$

26) $\frac{8x}{\cos^2(4x^2+3)}$ 27) $\frac{-1}{\sin^2\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}=\ldots$ 28) $2\sin\frac{1}{x}\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)$ 29) $\frac{\sin\sqrt[3]{1-x}\cdot\cos^2\sqrt[3]{1-x}}{\sqrt[3]{(1-x)^2}}$

30) $4\tan^3(1-e^x)\cdot\frac{1}{\cos^2(1-e^x)}\cdot(-e^x)$ 31) $5\cot^4(\ln x)\cdot\frac{-1}{\sin^2\ln x}\cdot\frac{1}{x}$ 32) $-\frac{1}{(\sin x)^2}$

33) 2·(-2)·(cos3x)$^{-3}$·(- sin3x) · 3 = $\frac{12\sin 3x}{(\cos 3x)^3}$ 34) $-\sqrt{\frac{1+x}{1-x}}\cdot\frac{1}{(1+x)^2}$

35) $\frac{1}{2\sqrt{2-t}}(-1)(11t^2+12)^{13}+13(11t^2+12)^{12}\cdot 22t\cdot\sqrt{2-t}=\ldots$ 36) $\frac{16-2t^2}{(16-t^2)\sqrt{16-t^2}}$

37) 3sin$^2t^2$·cost^2·2t = 6t sin$^2t^2$·cost^2 38) $\frac{-2}{2\sqrt{\cos 3x}}\cdot(-\sin 3x)\cdot 3=\frac{3\sin 3x}{\sqrt{\cos 3x}}$ 39) $\cos e^{2\varphi}\cdot e^{2\varphi}\cdot 2$

40) $-e^{\sin\varphi}\cdot\cos\varphi\cdot\cos\varphi+\sin\varphi\cdot e^{\sin\varphi}=\ldots$ 41) $\frac{1}{1+e^t}$ 42) $\frac{2}{\sqrt{1-4x^2}}$ 43) $\frac{-2}{\sqrt{1-\frac{t^2}{256}}}\cdot\frac{1}{16}=\ldots$

44) $\frac{1}{1+15t}\cdot\frac{1}{2\sqrt{15t}}\cdot 15$ 45) $\frac{-14}{1+\frac{169}{t^2}}\cdot\left(-\frac{13}{t^2}\right)$ 46) $\frac{1}{2\sqrt{\arcsin x^2}}\cdot\frac{1}{\sqrt{1+x^4}}\cdot 2x$

Example 5. Find the derivative of the function $y=(1+3x-5x^2)^{30}$ at point $x=0$.

Solution: Differentiating, we get $y'=30(1+3x-5x^2)^{29}\cdot(3-10x)$. Putting $x=0$ we obtain y' = 90.

1.2.2. Higher order derivatives of explicit function

A derivative of the second order or the second derivative of the function $y=f(x)$ is the derivative of its derivative, i.e., $y''=(y')'$.

In general, the n-th derivative of a function $y = f(x)$ is the derivative of a derivative of order $(n-1)$. We use for the notation of the n-th derivative $y^{(n)}$ or $f^{(n)}(x)$.

Example 6. Find the second derivative of the function $y=\ln(1-x)$.

Solution:

$$y'=-\frac{1}{1-x},\qquad y''=\left(-\frac{1}{1-x}\right)'=[-(1-x)^{-1}]'=(-1)\cdot(-1)\cdot(1-x)^{-2}=\frac{1}{(1-x)^2}$$

Exercises. Find the second derivatives of the functions:

47) $y=e^{x^2}$
48) $y=\sin^2 x$
49) $y=\arcsin^2 x$
50) $y=\sqrt{1-x}$
51) Find y''', if $y=\ln(\sin x)$
52) Find y^V of the function $y=\ln(1+x)$..
53) Find y^{VI} of the function $y=\sin 2x$.
54) Find $y^{(n)}$ $y = e^{kx}$ (k =const)
55) Find $y''(1)$ if $y=\frac{1}{3x+2}$,
56) Find $f'''(3)$, if $f(x)=(2x-3)^5$
57) Find $\ddot{h}(1)$ if $h(t)=(t^2-1)^2$

Answers.

47) $2e^{x^2}(2x^2+1)$ 48) $2(\cos^2 x-\sin^2 x)$ 49) $\frac{2}{1-x^2}+\frac{2x\cdot\arcsin x}{\sqrt{(1-x^2)^3}}$ 50) $-\frac{1}{4(1-x)\sqrt{1-x}}$. 51) $y'''=\frac{2\cos x}{\sin^3 x}$
52) $y^V=\frac{24}{(x+1)^5}$ 53) $y^{VI}=-64\sin 2x$ 54) $y^{(n)} = k^n e^{kx}$ 55) $y''(1)=\frac{18}{125}$ 56) $f'''(3)=4320$

57) 8

Example 7. Find the derivative y'' of the function $y=(1+3x)^3$ at point $x=-1$.

Solution: Differentiating, we get $y'=9(1+3x)^2$ ja $y''=54(1+3x)$. Putting $x=-1$ we obtain y' $=-108$.