2. Logarithmic Derivative

A logarithmic derivative of a function $y = f(x)$ is the derivative of the logarithm of the function, i.e.

$$(\ln y)' = \frac{y'}{y} = \frac{f'(x)}{f(x)}.$$

Finding the derivative is sometimes simplified by first taking the logarithm of the function.

Example. Find the derivative of the exponential function $y = u^v$, where $u = u(x)$ and $v = v(x)$.

Solution: Taking logarithms, we get $\ln y = v \cdot \ln u$ (by formula $\log_a c^k = k \cdot \log_a c$). Differentiate both sides of this equation with respect to x:

$$(\ln y)' = v' \ln u + v(\ln u)' \text{ or } \frac{y'}{y} = v' \ln u + v\frac{u'}{u}$$

whence $y' = y\left(v' \ln u + \frac{v}{u}u'\right)$ or $y' = u^v\left(v' \ln u + \frac{v}{u}u'\right)$.

Example. Find y' if $y = (1 - 2x)^{\sin x}$ tuletis.
Solution:
$\ln y = \ln (1 - 2x)^{\sin x}$
$\ln y = \sin x \cdot \ln (1 - 2x)$
$\frac{1}{y} \cdot y' = (\sin x)' \cdot \ln(1-2x) + (\ln(1-2x))' \cdot \sin x$
$\frac{1}{y} \cdot y' = \cos x \cdot \ln(1-2x) + \frac{1}{1-2x}(-2) \cdot \sin x$
$y' = \left[\cos x \cdot \ln(1-2x) - \frac{2}{1-2x} \cdot \sin x\right] \cdot (1-2x)^{\sin x}$

Example. Find y', if $y = \sqrt[3]{x^2}\,\frac{1-x}{1+x^2}\sin^3 x \cos^2 x$

Solution:
$\ln y = \ln\left(\sqrt[3]{x^2}\,\frac{1-x}{1+x^2}\sin^3 x \cos^2 x\right)$

Using the formulae $\log_a m \cdot n = \log_a m + \log_a n$ and $\log_a \frac{m}{n} = \log_a m - \log_a n$ one has
$\ln y = \frac{2}{3}\ln x + \ln(1-x) - \ln(1+x^2) + 3\ln \sin x + 2\ln \cos x$
$\frac{1}{y} \cdot y' = \frac{2}{3} \cdot \frac{1}{x} + \frac{-1}{1-x} - \frac{2x}{1+x^2} + 3\frac{1}{\sin x} \cdot \cos x - 2\frac{1}{\cos x} \cdot \sin x$
$y' = y\left(\frac{2}{3x} - \frac{1}{1-x} - \frac{2x}{1+x^2} + 3\cot x - 2\tan x\right)$

Example. Find y', if $y = (\sin x)^x$.

Solution:
$\ln y = \ln(\sin x)^x$
$\ln y = x \cdot \ln \sin x$
$\frac{1}{y} \cdot y' = \ln \sin x + x \cdot \frac{1}{\sin x} \cdot \cos x$
$y' = y(\ln \sin x + x \cdot \cot x) = (\sin x)^x (\ln \sin x + x \cdot \cot x).$

Exercises. Find the derivatives of the presented functions

1) $y = x^x$ 2) $y = (2x)^{1-x}$ 3) $y = (1-x)^{2x}$ 4) $y = (\sin x)^{\cos x}$ 5) $y = (\tan x - 1)^{x^2}$

6) $y = \left[\ln\left(4 - x^2\right)\right]^{\sqrt{x}}$ 7) $y = (\arcsin x)^{e^{\sin x}}$ 8) $y = x^{x^2}$ 9) $y = \sqrt[x]{x}$, 10) $y = x^{\sqrt{x}}$

11) $y = x^{x^x}$ 12) $y = x^{\sin x}$ 13) $y = (\cos x)^{\sin x}$ 14) $y = \left(1 + \frac{1}{x}\right)^x$ 15) $y = (\arctan x)^x$.

16) $y = (x+1)(2x+1)(3x+1)$ 17) $y = \frac{(x+2)^2}{(x+1)^3(x+3)^4}$, 18) $y = \sqrt{\frac{x(x-1)}{x-2}}$, 19) $y = x \cdot \sqrt[3]{\frac{x^2}{x^2+1}}$,

20) $y = \frac{(x-2)^9}{\sqrt{(x-1)^5(x-3)^{11}}}$, 21) $y = \frac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2 \cdot \sqrt{(x+3)^3}}}$,

Answers.

1) $[\ln x + 1] \cdot x^x$ 2) $\left[-\ln 2x + \frac{1-x}{x}\right] \cdot (2x)^{1-x}$ 3) $\left(2\ln(1-x) - \frac{2x}{1-x}\right) \cdot (1-x)^{2x}$

4) $\left[-\sin x \ln \sin x + \frac{\cos^2 x}{\sin x}\right] \cdot (\sin x)^{\cos x}$ 5) $\left[2x\ln(\tan x - 1) + \frac{x^2}{(\tan x - 1)\cos^2 x}\right] \cdot (\tan x - 1)^{x^2}$

6) $\left[\frac{\ln\left(4-x^2\right)}{2\sqrt{x}} - \frac{2x\sqrt{x}}{4-x^2}\right] \cdot \left[\ln\left(4-x^2\right)\right]^{\sqrt{x}}$ 7) $\left[e^{\sin x}\cos x \ln(\arcsin x) + \frac{e^{\sin x}}{\arcsin x \cdot \sqrt{1-x^2}} \cdot e^{\sin x}\right]$,

8) $x^{x^2+1}(1 + 2\ln x)$, 9) $\sqrt[x]{x} \cdot \frac{1-\ln x}{x^2}$, 10) $x^{\sqrt{x}-\frac{1}{2}}\left[1 + \frac{\ln x}{2}\right]$, 11) $x^{x^x} \cdot x^x\left(\frac{1}{x} + \ln x + \ln^2 x\right)$,

12) $\left[\frac{\sin x}{x} + \cos x \ln x\right] \cdot x^{\sin x}$, 13) $\left[\cos x \ln \cos x - \frac{\sin^2 x}{\cos x}\right] \cdot (\cos x)^{\sin x}$ 14) $\left[\ln\left(1 + \frac{1}{x}\right) + \frac{1}{1+x}\right] \cdot \left(1 + \frac{1}{x}\right)^x$,

15) $\left[\ln \arctan x + \frac{x}{(1+x^2)\arctan x}\right] \cdot (\arctan x)^x$ 16) $(1+2x)(1+3x) + 2(1+x)(1+3x) + 3(1+x)(1+2x)$,

17) $-\frac{(x+2)(5x^2+19x+20)}{(x+1)^4(x+3)^5}$, 18) $\frac{x^2-4x+2}{2\sqrt{x(x-1)(x-2)^3}}$, 19) $\frac{3x^2+5}{3(x^2+1)} \cdot \sqrt[3]{\frac{x^2}{x^2+1}}$, 20) $\frac{(x-2)^9(x^2-7x+1)}{(x-1)(x-2)(x-3)\sqrt{(x-1)^5(x-3)^4}}$,

21) $-\frac{5x^2+x-24}{3\sqrt{(x-1)(x+3)^5}\sqrt[3]{(x+2)^5}}$,