3. Derivative of an implicit function

If the relationship between x and y is given implicitly

$$F(x, y) = 0, \qquad (1)$$

then, in order to find the derivative $y'_x = y'$ in the simplest cases, it is sufficient

1) to calculate the derivative, with respect to x, of the left hand side of (1), taking y as a function of x,
2) to equate this derivative to zero, i.e., to set $\frac{d}{dx}F(x, y) = 0$.
3) to solve the resulting equation for y'.

Example. Find the derivative y' if $x^2 + y^2 - 3axy = 0$.

Solution: Forming the derivative of the left hand side of the given function and equating it to zero, we get

$$3x^2 - 3y^2y' - 3a(y + xy') = 0,$$

whence

$$y' = \frac{x^2 - ay}{ax - y^2}$$

Example. Find y' at the point $M(1,1)$, if $2y = 1 + xy^3$.

Solution: Differentiating, we get $2y' = y^3 + 3xy^2y'$. Putting $x = 1$ and $y = 1$, we obtain $2y' = 1 + 3y'$, whence $y' = -1$.

Exercise 1. Find the derivative x'_y if

$$a)\ y = 3x + x^2, \qquad b)\ y = x - \frac{1}{2}\sin x, \qquad c)\ y = 0{,}1x + e^{\frac{x}{2}}.$$

Exercise 2. Find the derivative $y' = dy/dx$ of implicit functions y.

2.1. $2x - 5y + 10 = 0$ **2.2.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ **2.3.** $x^3 + y^3 = a^3.$ **2.4.** $x^3 + x^2y + y^2 = 0.$

2.5. $\sqrt{x} + \sqrt{y} = \sqrt{a}.$ **2.6.** $\sqrt[3]{x^2} + \sqrt[3]{y^2} = \sqrt[3]{a^2}.$ **2.7.** $y^3 = \frac{x - y}{x + y}.$ **2.8.** $y - 0.3\sin y = x.$

2.9. $a\cos^2(x + y) = b.$ **2.10.** $\tan y = xy.$ **2.11.** $xy = \arctan\frac{x}{y}.$ **2.12.** $\arctan(x + y) = x.$

2.13. $e^y = x + y.$ **2.14.** $\ln x + e^{-\frac{y}{x}} = c.$ **2.15.** $\ln y + \frac{x}{y} = c.$ **2.16.** $\arctan\frac{y}{x} = \frac{1}{2}\ln(x^2 + y^2).$

Exercise 3. Find the derivative y' of the specified functions y at the indicated points:

3.1. $(x + y)^3 = 27(x - y)$ for $x = 2$ and $y = 1$.

3.2. $ye^y = e^{x+1}$ for $x = 0$ and $y = 1$.

3.3. $y^2 = x + \ln\frac{y}{x}$ **4.1.** for $x = 1$ and $y = 1$.

Exercise 4. Find the derivative y'' of the functions represented implicity.

4.1. $y^2 = 2px.$ **4.2.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ **4.3.** $y = x + \arctan y$ **4.4.** $y = x + \ln y.$

Exercise 5. Having the equation $y = x + \ln y$, find y''_x and x''_y.

Exercise 6. Find the derivative y''' of the function $x^2 + y^2 = a^2$

Exercise 7. The function is defined implicitly. Find the given derivatives at the pont M.

7.1. $x^2+5xy+y^2-2x+y-6=0,\quad y'',\quad M(1;1).$

7.2. $x^4-xy+y^4=1,\quad y'',\quad M(0;1)$

7.3. $x^2+2xy+y^2-4x+2y-2=0,\quad y''',\quad M(1;1).$

Answers.

2.1. $\frac{2}{5}$. **2.2.** $-\frac{b^2x}{a^2y}$. **2.3.** $-\frac{x^2}{y^2}$. **2.4.** $-\frac{x(3x+2y)}{x^2+2y}$. **2.5.** $-\sqrt{\frac{y}{x}}$. **2.6.** $-\sqrt[3]{\frac{y}{x}}$. **2.7.** $\frac{1-y^3}{1+3xy^2+4y^3}$.
2.8. $\frac{10}{10-3\cos y}$. **2.9.** -1. **2.10.** $\frac{\mathrm{y}\cos^2 y}{1-\mathrm{x}\cos^2 y}$. **2.11.** $\frac{y(1-x^2-y^2)}{x(1+x^2+y^2)}$. **2.12.** $(x+y)^2$
2.13. $y'=\frac{1}{e^y-1}=\frac{1}{x+y-1}$. **2.14.** $\frac{y}{x}+e^{\frac{y}{x}}$. **2.15.** $\frac{y}{x-y}$. **2.16.** $\frac{x+y}{x-y}$. **3.1.** 0. **3.2.** $\frac{1}{2}$.
3.3. 0. **4.1.** $-\frac{p^2}{y^3}$. **4.2.** $-\frac{b^4}{a^2y^3}$. **4.3.** $-\frac{2y^2+2}{y^5}$. **4.4.** $\frac{y}{(1-y)^3}$. **5.** $y''_x=\frac{y}{(1-y)^3}$; $x''_y=\frac{1}{y^2}$.
6. $-\frac{3a^2x}{y^5}$. **7.1.** $\frac{111}{256}$. **7.2.** $-\frac{1}{16}$. **7.3.** $\frac{1}{3}$.