# 4. The derivative of parametrically represented function

## 4.1. The parametric definition of a curve

In the first example below we shall show how the $x$ and $y$ coordinates of points on a curve can be defined in terms of a third variable, $t$, the parameter.

**Example.** Consider the parametric equations

$$x = \cos t \quad y = \sin t \text{ for } 0 \le t \le 2\pi \qquad (1)$$

Note how both $x$ and $y$ are given in terms of the third variable $t$.

To assist us in plotting a graph of this curve we have also plotted graphs of $\cos t$ and $\sin t$ in Fig. 1. Clearly,

when $t = 0$, $x = \cos 0 = 1$; $y = \sin 0 = 0$

when $t = \frac{\pi}{2}$, $x = \cos\frac{\pi}{2} = 0$; $y = \sin\frac{\pi}{2} = 1$.

In this way we can obtain the $x$ and $y$ coordinates of lots of points given by equations (1). Some of these are given in Table 1.

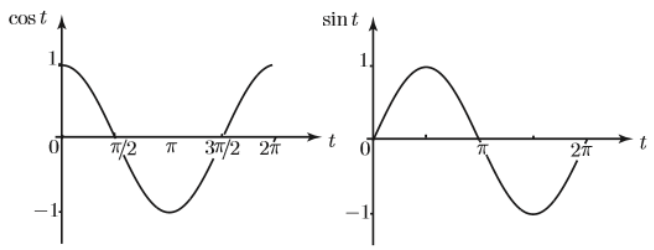
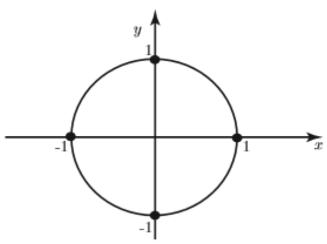



Figure 1. Graphs of $\sin t$ and $\cos t$.

| $t$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $x$ | 1 | 0 | −1 | 0 | 1 |
| $y$ | 0 | 1 | 0 | −1 | 0 |

Table 1. Values of $x$ and $y$ given by equations (1).

Plotting the points given by the x and y coordinates in Table 1, and joining them with a smooth curve we can obtain the graph. In practice you may need to plot several more points before you can be confident of the shape of the curve. We have done this and the result is shown in Figure 2.




So $x = \cos t$, $y = \sin t$, for $t$ lying between $0$ and $2\pi$, are the parametric equations which describe a circle, centre $(0,0)$ and radius $1$.

Figure 2. The parametric equations define a circle centered at the origin and having radius 1.

## 4.2 Differentiation of a function defined parametrically

If a function $y$ is related to an argument $x$ by means of a parameter $t$

$$\begin{cases} x = \varphi(t) \\ y = \theta(t) \end{cases} \quad \text{then} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

**Example.** Find $\frac{dy}{dx}$ if $\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$

*Solution:* We find $\frac{dx}{dt} = -a\sin t$, $\frac{dy}{dt} = a\cos t$, whence $\frac{dy}{dx} = -\frac{a\cos t}{a\sin t} = -\cot t$.

**Example.** Calculate $dy/dx$ when $t = \frac{\pi}{2}$, if $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$

Solution:

$$y' = \frac{a\sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t}, \quad y'|_{\frac{\pi}{2}} = \frac{\sin\frac{\pi}{2}}{1 - \cos\frac{\pi}{2}} = 1.$$

**Exercise 1.** In the following problems, find the derivative $y' = dy/dx$ of the functions $y$ represented parametrically:

**1.1** $\begin{cases} x = 2t - 1 \\ y = t^3 \end{cases}$ **1.2** $\begin{cases} x = \frac{1}{t+1} \\ y = \left(\frac{t}{t+1}\right)^2 \end{cases}$ **1.3** $\begin{cases} x = \frac{2at}{1+t^2} \\ y = \frac{a(1-t^2)}{1+t^2} \end{cases}$ **1.4** $\begin{cases} x = \dfrac{3at}{1+t^3} \\ y = \dfrac{3at^2}{1+t^3} \end{cases}$

**1.5** $\begin{cases} x = \sqrt{t} \\ y = \sqrt[3]{t} \end{cases}$ **1.6** $\begin{cases} x = \sqrt{1+t^2} \\ y = \frac{t-1}{\sqrt{1+t^2}} \end{cases}$ **1.7** $\begin{cases} x = a(\cos t + t\sin t) \\ y = a(\sin t - t\cos t) \end{cases}$ **1.8** $\begin{cases} x = a\cos^2 t \\ y = b\sin^2 t \end{cases}$

**1.9** $\begin{cases} x = a\cos^3 t \\ y = b\sin^3 t \end{cases}$ **1.10** $\begin{cases} x = \dfrac{\cos^3 t}{\sqrt{\cos 2t}} \\ y = \dfrac{\sin^3 t}{\sqrt{\cos 2t}} \end{cases}$ **1.11** $\begin{cases} x = \arccos\frac{1}{\sqrt{1+t^2}} \\ y = \arcsin\frac{t}{\sqrt{1+t^2}} \end{cases}$ **1.12** $\begin{cases} x = e^{-t} \\ y = e^{2t} \end{cases}$

**Exercise 2.** Find $y'$ when $t$ has a given value.

**2.1** $\begin{cases} x = t \cdot \ln t \\ y = \frac{\ln t}{t} \end{cases}, \ t = 1$ **2.2** $\begin{cases} x = e^t\cos t \\ y = e^t\sin t \end{cases}, \ t = \frac{\pi}{4}.$

## 4.3 Higher-order derivatives of parametrically represented functions

If $\begin{cases} x = \varphi(t) \\ y = \theta(t) \end{cases}$ then the derivatives $y'_x$ can successively be calculated by the formulae $y'_x = \frac{y'_t}{x'_t}$,

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This derivative can be rewritten parametrically

$$\begin{cases} x = \varphi(t) \\ y'_x = \dfrac{y'_t}{x'_t} = g(t), \end{cases}$$

or a second derivative, we have the formula

$$y''_{xx} = (y'_x)'_x = \frac{\left(\frac{y'_t}{x'_t}\right)'_t}{x'_t} = \frac{x'_t\, y''_{tt} - x_{tt}\, y'_t}{(x'_t)^3}.$$

The third derivative:

$$y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t}$$

**Example.** Find $y''$, if $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$

*Solution:* We have (sometime derivative by $t$ can be denoted as $\dot{y}$)

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t.$$

$$y'' = (y')' = \frac{(\dot{y'})}{\dot{x}} = \frac{\left(-\frac{b}{a}\cot t\right)}{-a\sin t} = \frac{-\frac{b}{a}\cdot\frac{-1}{\sin^2 t}}{-a\sin t} = -\frac{b}{a^2\sin^3 t}.$$

**Exercise 3.** Find $d^2y/dx^2$ for the following problems:

**3.1** $\begin{cases} x = \ln t \\ y = t^3 \end{cases}$ **3.2** $\begin{cases} x = \arctan t \\ y = \ln(1+t^2) \end{cases}$ **3.3** $\begin{cases} x = arcsin\, t \\ y = \sqrt{1-t^2} \end{cases}$ **3.4** $\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$ **3.5** $\begin{cases} x = a\cos^3 t \\ y = b\sin^3 t \end{cases}$

**3.6** $\begin{cases} x = a(t - \sin t) \\ y = a(1-\cos t) \end{cases}$ **3.7** $\begin{cases} x = a(\sin t - t\cos t) \\ y = a(\cos t - t\sin t) \end{cases}$ **3.8** $\begin{cases} x = \cos 2t \\ y = \sin^2 t \end{cases}$ **3.9** $\begin{cases} x = e^{-at} \\ y = e^{at} \end{cases}$ **3.10** $\begin{cases} x = arctan\, t \\ y = \frac{1}{2}t^2 \end{cases}$

**3.11** $\begin{cases} x = \ln t \\ y = \frac{1}{1-t} \end{cases}$

**Exercise 4.** Find $y''$ when t has given value.

**4.1** $\begin{cases} x = \ln(1+t^2) \\ y = t^2 \end{cases}$, $y''$, $t = 0$ **4.2** $\begin{cases} x = t^2 \\ y = 2\cos t \end{cases}$, $y''$, $t = \frac{\pi}{2}$

**Example.** Find $y'''$, if $\begin{cases} x = e^{-t} \\ y = t^3 \end{cases}$

*Solution:* We have

$$y'_x = \frac{3t^2}{-e^{-t}} = -3t^2e^t$$

and

$$\begin{cases} x = e^{-t} \\ y'_x = -3t^2e^t \end{cases} \qquad y''_{xx} = \left(y'_x\right)'_x = \frac{(-3t^2e^t)'}{(e^{-t})'} = -\frac{6te^t + 3t^2e^t}{-e^{-t}} = e^{2t}(6t + 3t^2).$$

$$\begin{cases} x = e^{-t} \\ y'_x = e^{2t}(6t+3t^2) \end{cases} \qquad y'''_{xxx} = (y''_{xx})'_x = \frac{(e^{2t}(6t+3t^2))'}{(e^{-t})'} = \frac{2e^{2t}(6t+3t^2) + e^{2t}(6+6t)}{-e^{-t}}$$
$$= -6e^{3t}(t^2 + 3t + 1).$$

**Exercise 5.** Find $d^3y/dx^3$ for the following problems:

**5.1** $\begin{cases} x = \frac{1}{\cos^2 x} \\ y = \tan x \end{cases}$ **5.2** $\begin{cases} x = e^{-t}\cos t \\ y = e^{-t}\sin t \end{cases}$ **5.3** $\begin{cases} x = e^{-t} \\ y = t^3 \end{cases}$

**Answers.**

**1.1** $\frac{2}{3}t^2$. **1.2** $\frac{t-1}{t+1}$. **1.3** $-\frac{2t}{1-t^2}$. **1.4** $\frac{t(2-t^3)}{1-2t^3}$. **1.5** $\frac{2}{3\sqrt[6]{t}}$. **1.6** $\frac{t+1}{t(t^2+1)}$. **1.7** $\tan t$. **1.8** $-\frac{b}{a}$. **1.9** $-\frac{b}{a}\tan t$.
**1.10** $-\tan 3t$. **1.11** $-2e^{3t}$ **1.12** $\tan t$. **2.1.** $1$. **2.2** $\infty$. **3.1** $9t^3$. **3.2** $2t^2 + 2$. **3.3** $-\sqrt{1-t^2}$.
**3.4** $-\frac{1}{a\sin^3 t}$. **3.5** $\frac{1}{3a\cos^4 t\sin t}$. **3.6** $-\frac{1}{4a\sin^4\frac{t}{2}}$. **3.7** $-\frac{1}{at\sin^3 t}$. **3.8** $0$. **3.9** $2e^{3at}$. **3.10** $(1+t^2)(1+3t^2)$.
**3.11** $\frac{t(1+t)}{(1-t)^3}$. **4.1** $1$. **4.2** $\frac{4}{\pi^3}$ **5.1** $\frac{2\cot^4 t}{\sin t}$. **5.2** $\frac{2e^{2t}(2\sin t - \cos t)}{(\sin t + \cos t)^5}$. **5.3** $-6e^{3t}(1 + 3t + t^2)$.