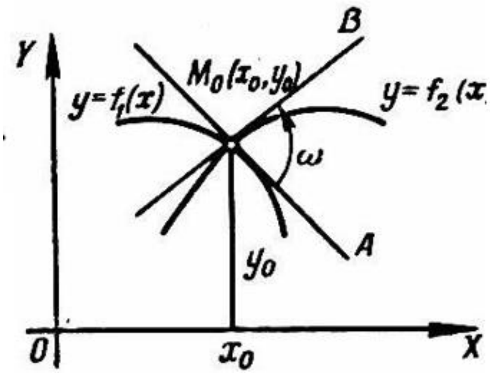
6.2 Angle between curves

The angle between the curves $y = f_1$, $y = f_2$ at their common point $M_0(x_0, y_0)$ (see Fig. 1) is the angle ω between the tangents M_0A and M_0B to these curves at the point M_0.

Using a familiar formula of analytic geometry, we find

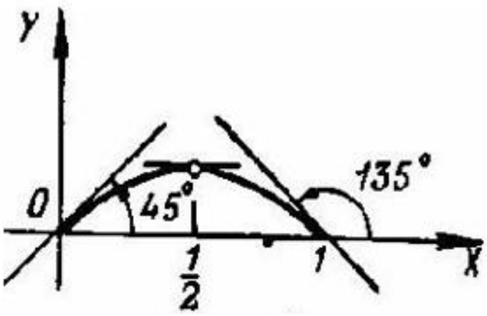
$$\tan \omega = \frac{f_2{}'(x_0) - f_1{}'(x_0)}{1 - f_1{}'(x_0) \cdot f_2{}'(x_0)}$$



Example 1. What angles are formed with the x —axis by the tangents to the curve $y = x - x^2$ at the points with the abscise: a) x=0, b) x=1/2, c) x=1?

Solution. We have $y' = 1 - 2x$, whence

a) $\tan \omega = 1, \ \ \omega = 45°$
b) $\tan \omega = 0, \ \ \omega = 0°$
c) $\tan \omega = -1, \ \ \omega = 135°$



Example 2. Find the angle of intersection between the two curves: $xy + y = 1$ and $y^3 = (x+1)^2$

Solution: Consider the given pair of two curves above. The first thing that we need to do is to get their point of intersection by solving the systems of equation as follows

$$\begin{cases} xy + y = 1 \\ y^3 = (x+1)^2 \end{cases} \to \begin{cases} y = \frac{1}{x+1} \\ y^3 = (x+1)^2 \end{cases} \to \begin{cases} y = \frac{1}{x+1} \\ \left(\frac{1}{x+1}\right)^3 = (x+1)^2 \end{cases} \to \begin{cases} y = \frac{1}{x+1} \\ (x+1)^5 = 1 \end{cases} \to \begin{cases} x = 0 \\ y = 1 \end{cases}$$

The point of intersection of the given two curves is P(0, 1).
The slope of a curve is equal to the first derivative of the equation of a curve with respect to x. In this case, dy/dx is the slope of a curve.

Consider the first given equation of a curve $xy + y = 1$. Take the derivative on both sides of the equation with respect to x by implicit differentiation, we have

$$\frac{d(xy + y = 1)}{dx} \to x\frac{dy}{dx} + y + \frac{dy}{dx} = 0 \to \frac{dy}{dx}(x+1) = -y \to \frac{dy}{dx} = -\frac{y}{x+1}$$

To get the value of the slope of a curve at their point of intersection, substitute $x = 0$ and $y = 1$ at the equation above, we have

$$\left.\frac{dy}{dx}\right|_{\substack{x=0,\\ y=1}} = -\frac{1}{0+1} = -1.$$

The slope of a curve at their point of intersection is equal to the slope of tangent line that passes thru also at their point of intersection $f_1 = -1$.

Consider the second given equation of a curve $y^3 = (x+1)^2$. Take the derivative on both sides of the equation with respect to x by implicit differentiation, we have

$$\frac{d(\,y^3 = (x+1)^2)}{dx} \;\rightarrow\; 3y^2\frac{dy}{dx} = 2(x+1) \;\rightarrow\; \frac{dy}{dx} = \frac{2(x+1)}{3y^2}$$

To get the value of the slope of a curve at their point of intersection, substitute x = 0 and y = 1 at the equation above, we have

$$\left.\frac{dy}{dx}\right|_{\substack{x=0,\\ y=1}} = \frac{2(0+1)}{3(1)^2} = \frac{2}{3}$$

The slope of a curve at their point of intersection is equal to the slope of tangent line that passes thru also at their point of intersection $f_2 = \frac{2}{3}$.

Therefore, the angle between two curves at their point of intersection is

$$\tan\omega = \frac{f_2{}'(x_0) - f_1{}'(x_0)}{1 - f_1{}'(x_0)\cdot f_2{}'(x_0)} = \frac{\frac{2}{3} - (-1)}{1 + (-1)\cdot\frac{2}{3}} = 5$$

$\omega = \arctan 5 \approx 78{,}7°$ or $180° - 78{,}7° = 101{,}3°$

Example 3. Find the angle of intersection between the two curves: $y = 2x$, $x^5 + y^5 = 33$

Solution: The first thing that we need to do is to get their point of intersection by solving the systems of equation as follows

$$\begin{cases} y = 2x \\ x^5 + y^5 = 33 \end{cases} \rightarrow \begin{cases} y = 2x \\ x^5 + (2x)^5 = 33 \end{cases} \rightarrow \begin{cases} y = 2x \\ x^5 + 32x^5 = 33 \end{cases} \rightarrow \begin{cases} y = 2x \\ 33x^5 = 33 \end{cases} \rightarrow \begin{cases} y = 2x \\ x = 1 \end{cases} \rightarrow \begin{cases} y = 2 \\ x = 1 \end{cases}$$

The point of intersection of the given two curves is P(1, 2).

The slope of a curve is equal to the first derivative of the equation of a curve with respect to x. In this case, dy/dx is the slope of a curve. Actually, the first curve is a straight line and since the right side of the equation contains coefficient only, the slope of the first curve is $f_1 = 2$

Consider the second given equation of a curve. The slope of a curve is equal to the first derivative of the equation of a curve with respect to x. In this case, dy/dx is the slope of a curve. Take the derivative on both sides of the equation with respect to x by implicit differentiation, we have

$$\frac{d(\,x^5 + y^5 = 33)}{dx} \;\rightarrow\; 5y^4\frac{dy}{dx} = -5x^4 \;\rightarrow\; \frac{dy}{dx} = -\frac{x^4}{y^4}$$

To get the value of the slope of a curve at their point of intersection, substitute x = 1 and y = 2 at the equation above, we have

$$\left.\frac{dy}{dx}\right|_{\substack{x=0,\\ y=1}} = -\frac{1^4}{2^4} = -\frac{1}{16}$$

The slope of a curve at their point of intersection is equal to the slope of tangent line that passes thru also at their point of intersection $f_2 = -\frac{1}{16}$.

Therefore, the angle between two curves at their point of intersection is

$$\tan\omega = \frac{f_2'(x_0) - f_1'(x_0)}{1 - f_1'(x_0)\cdot f_2'(x_0)} = \frac{-\frac{1}{16} - 2}{1 + 2\cdot\left(-\frac{1}{16}\right)} = -\frac{33}{14}$$

$\omega = \arctan 5 \approx 67°$ or $180° - 67° = 113°$

Exercises

1. At what angles do the curves y= sin x and *y* = sin 2x intersect the abscissae at the origin?
2. At what angle does y = tan x intersect the abscissa at the origin?
3. At what angle does the curve $y=e^{0.5x}$ intersect the straight line x=2?
4. Find the angle at which the parabolas $y= (x - 2)^2$ and $y = -4 + 6x - x^2$ intersect.
5. At what angle do the parabolas $y=x^2$ and $y = x^3$ intersect?

Answers.

1. $45°; \arctan 2 \approx 63°26'$ **2.** $45°$ **3.** $\arctan\frac{2}{e} \approx 36°21'$ **4.** $40°36'$ **5.** The parabolas are tahgent at the point (0,0) and intersect at an angle $\arctan\frac{1}{7} \approx 8°8'$ at the point (1,1).