4 Geometry and Mechanics Applications of the Derivative

4.1 Equations of the tangent and the normal: It follows from the geometric significance of the derivative that the equation of the tangent to a curve $y = f(x)$ or $F(x, y) = 0$ at a point $M(x_0, y_0)$ will be

$$y - y_0 = y_0'(x - x_0),$$

where y_0' is the value of the derivative y' at the point $M(x_0, y_0)$. The straight line passing through the point where the tangent touches the curve, perpendicularly to the tangent, is called the normal to the curve. The normal has the equation

$$x - x_0 + y_0'(y - y_0) = 0.$$

4.2 Angle between curves The angle between the curves

$$y = f_1(x), \; y = f_2(x)$$

at their common point $M_0(x_0, y_0)$ (Fig. 12) is the angle ω between the tangents M_0A and M_0B to these curves at the point M_0.

Using a familiar formula of analytic geometry, we find

$$\tan\omega = \frac{f_2'(x_0) - f_1'(x_0)}{1 + f_1'(x_0) \cdot f_2'(x_0)}.$$

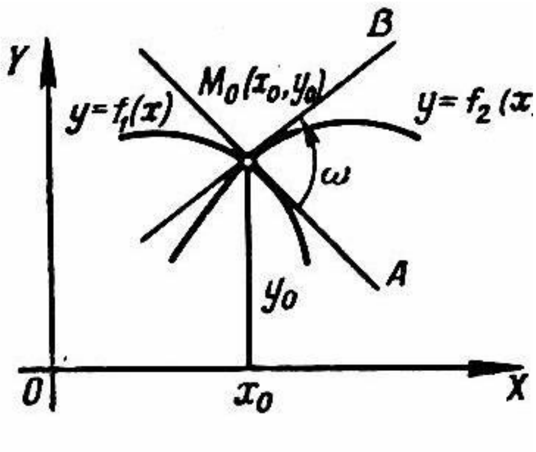

Fig. 12

4.3 Segments associated with the tangent and the normal in an orthogonal co-ordinate system: The tangent and the normal determine the four segments:

$t = TM$ is the *segment of the tangent*,
$S_t = TK$ is the *subtangent*,
$n = NM$ is the *segment of the normal*,
$S_n = KN$ is the *subnormal*.

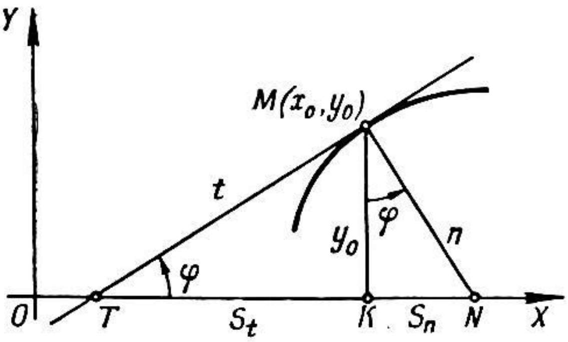

Fig. 13

Since $KM = |y_0|$ and $\tan\alpha = y_0'$, It follows that

$$t = TM = \left|\frac{y_0}{y_0'}\sqrt{1 + (y_0')^2}\right|; \quad n = NM = \left|y_0\sqrt{1 + (y_0')^2}\right|; \quad S_t = TK = \left|\frac{y_0}{y_0'}\right|; \quad S_n = |y_0 y_0'|$$

4.4 Segments associated with the tangent and the normal in a polar co-ordinate system: If a curve is given in polar co-ordinates by the equation $r = f(\varphi)$, then the angleμ formed by the tangent MT and the radius vector $r = OM$ (Fig.14) is defined by

$$\tan\mu = r\frac{d\varphi}{dr} = \frac{r}{r'}.$$

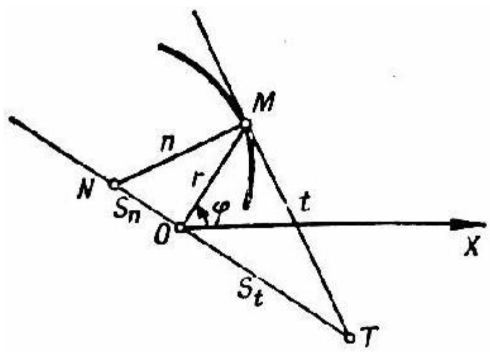

Fig. 14

The tangent *MT* and the normal *MN* at the point *M* together with the radius s vector of the point of tangency and with the perpendicular to the radius vector drawn through the pole O determine the following four segments (Fig. 14):

$t = MT$ is the *segment of the polar tangent*,
$n = MN$ is the *segment of the polar normal*,
$S_t = OT$ is the *polar subtangent*,
$S_n = ON$ is the *polar subnormal*.

These segments are given by the formulae:

$$t = MT = \frac{r}{|r'|}\sqrt{r^2 + (r')^2}; \qquad S_t = OT = \frac{r^2}{|r'|};$$
$$n = MN = \sqrt{r^2 + (r')^2}; \qquad S_n = ON = |r'|.$$

Exercises 621 - 666

621: What angles are formed with the x-axis by the tangents to the curve $y = x - x^2$ at the points with the abscise: a) x=0, b) x=1/2, c) x=1?

Solution. We have y' = l - 2x, whence

a) $\tan\varphi = 1, \varphi = 45°$; b) $\tan\varphi = 0, \varphi = 0°$;
c) $\tan\varphi = -1, \varphi = 135°$ (Fig. 15).

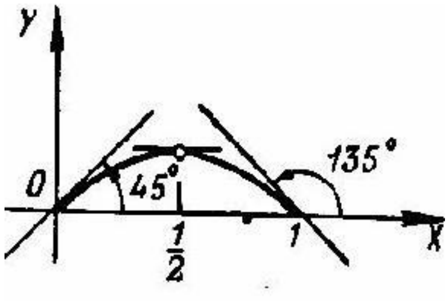

Fig. 15

622: At what angles do the curves y= sin x and *y* = sin 2x intersect the abscissae at the origin?

623: At what angle does y = tan x intersect the abscissa at the origin?

624: At what angle does the curve $y=e^{0.5x}$ intersect the straight line x=2?

625. Find the points at which the tangents to the curve $y = 3x^4 + 4x^3 - 12x^2 + 20$ are parallel to the *x*-axis.

626. At what point are the tangent to the parabola $y = x^2 - 7x = 3$ and the straight line 5*x* + *y* - 3 = 0 parallel?

627. Find the equation of the parabola $y = x^2 + bx + c$ which is tangent to the straight line *x* = *y* at the point (1,1).

628. Determine the slope of the tangent to the curve $x^3 + y^3 - xy - 7 = 0$ at the point (1, 2).

629. At what point of the curve $y^2 = 2x^3$ is the tangent perpendicular to the straight line 4*x* - 3y + 2 = 0?

630. Write the equation of the tangent and the normal to the parabola $y = \sqrt{x}$ at the point with abscissa *x* = 4.

Solution: We have $y' = \frac{1}{2\sqrt{x}}$, whence the slope of the tangent *is k* = $|y'|_{x=4}$ = 1/4. Since the point of the tangent has the co-ordinates x = 4, y = 2, the equation of the tangent is *y* - 2 = l/4(x - 4) or

$$x - 4y + 4 = 0$$

Since the slope of the normal must be perpendicular, $k_1 = -4$, whence the equation of the normal is

$$y - 2 = -4(x - 4) \quad or \quad 4x + y - 18 = 0.$$

631. Write the equations of the tangent and the normal to the curve $y = x^3 + 2x^2 - 4x - 3$ at the point (-2,5).

632. Find the equations of the tangent and the normal to the curve $y = \sqrt[3]{x-1}$ at the point (1,0).

633. Form the equations of the tangent and the normal to the curves at the given points:

a) *y* = tan 2*x* at the origin;
b) y = arsin[(x - 1)/2] at the intersection with the x-axis;
c) y = arcos 3x at the intersection with the y-axis;
d) *y* = ln *x* at the intersection with the .x-axis;
e) $y = e^{1-x^2}$ at the intersection with the straight line *y* = 1

634. Write down the equations of the tangent and the normal to the curve $x = \frac{1+t}{t^3},\ y = \frac{3}{2t^2} + \frac{1}{2t}$ at the point (2,2) .

635. Find the equations of the tangent to the curve $x = t\ cost, y = t\ sint$ at the origin and the point t = $\pi/4$.

636. Find the equations of the tangent and the normal to the curve $x^3 + y^2 + 2x - 6 = 0$ at the point with ordinate *y* = 3.

637. Find the equation of the tangent to the curve $x^5 + y^5 - 2xy = 0$ at the point (1,1).

638. Find the equations of. the tangents and normals to the curve *y* = (*x* - 1)(*x* - 2) (*x* - 3) at its intersection with the x-axis.

639. Find the equations of the tangent and the normal to the curve $y^4 = 4x^4 + 6xy$ at the point (1,2).

640*. Show that the segment of the tangent to the hyperbola $xy = a^2$ (the segment lies between the co-ordinate axes) is divided in two at the point of tangency.

641. Show that in the case of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ the segment of the tangent between the co-ordinate axes has the .constant value a.

642. Show that the normals to the involute of the circle $x = a(\cos t + t\sin t),\ \ y = a(\sin t - t\cos t)$ are tangents to the circle $x^2 + y^2 = a^2$.

643. Find the angle at which the parabolas $y= (x - 2)^2$ and $y = -4 + 6x - x^2$ intersect.

644. At what angle do the parabolas $y=x^2$ and $y = x^3$ intersect?

645. Show that the curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 10$ are tangent to each other at the point (3, 34). Will we have the same thing at (-2, 4)?

646. Show that the hyperbolas $xy = a^2, x^2 - y = b^2$ intersect at a right angle.

647. Given a parabola $y^2 = 4x$, evaluate at the point (1,2) the lengths of the segments of the sub-tangent, sub-normal, tangent and normal.

648. Find the length of the segment of the sub-tangent of the curve $y = 2^x$ at any of its points.

649. Show that in the equi-lateral hyperbola $x^2 - y^2 = a^2$ the length of the normal at any point is equal to the radius vector of that point.

650. Show that the length of the segment of the subnormal in the hyperbola $x^2 - y^2 = a^2$ at any point is equal to the abscissa of this point.

651. Show that the segments of the sub-tangents of the ellipse $x^2/a^2 - y^2/b^2 = 1$ and the circle $x^2 + y^2 = a^2$ at points with the same abscissas are equal. What procedure of construction of the tangent to the ellipse follows from this?

652. Find the length of the segment of the tangent, the normal, the sub-tangent and the sub-normal of the cycloid

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$$

at an arbitrary point $t = t_0$.

653. Find the angle between the tangent and the radius vector of the point of tangency in the case of the logarithmic spiral$r = ae^{k\varphi}$.

654. Find the angle between the tangent and the radius vector of the point of tangency for the lemniscate$r^2 = a^2 cos2\varphi$.

655. Find the lengths of the segments of the polar sub-tangent, sub-normal, tangent and normal as well as the angle between the tangent and the radius vector of the point of tangency in the case of the spiral of Archimedes$r = a\varphi$at the point with the polar angle $\varphi = 2\pi$.

656. Find the lengths of the segments of the polar sub-tangent, sub-normal, tangent and normal as well as the angle between the tangent and the radius vector of the hyperbolic spiral $r = a/\varphi$ at an arbitrary point $\varphi = \varphi_0,\ r = r_0 +$.

657. The law of motion of a point on the x-axis is$x = 3t - t^3$. Find the velocity of the point at t_0= 0, t_1= 1, t_2 =2 (*x* is in centimetres and *t* in seconds).

658. Two points move along the x-axis with the laws of motion$x = 100 + 5t,\ x = \frac{1}{2t^2}$, where $t \geq 0$. At what speed are these points receding from each other at the time of encounter (*x* is in centimetres, *t* is in seconds)?

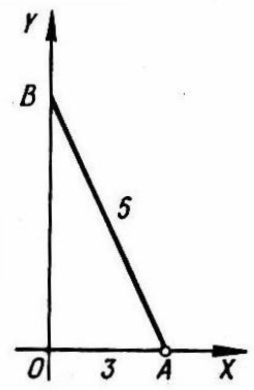

Fig. 16

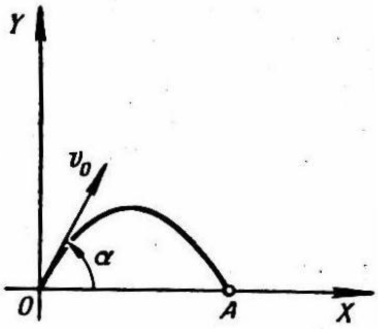

Fig. 17

659. The end-points of a segment *AB* = 5 *m* are sliding along t:he co-ordinate axes *OX* and *OY* (Fig. 16), *A* is moving at 2 m/sec. What is the velocity of *B* when *A* is at a distance OA = 3 m from the origin?

660*. The law of motion of a material point thrown up at an angle α to the horizontal with initial velocity v_0 (in the vertical plane *OXY* in Fig. 17 is given by the formulae (air resistance being neglected):$x = v_0 t cos\ \alpha,\ y = v_0 t sin\alpha - \frac{gt^2}{2}$, where *t* is the time and g is the acceleration of gravity. Find the trajectory of motion and the distance covered. Moreover, determine its velocity and its direction of motion.

661. A point is in motion along the hyperbola y = 10/x so that its abscissa *x* increases uniformly at a rate of 1 unit per second. What is the rate of change of its ordinate when the point passes through (5,2)?

662. At what point of the parabola $y^2 = 18x$ does the ordinate increase at twice the rate of the abscissa?

663. One side of a rectangle, a = 10 cm, is of constant length, while the other side b increases at a constant rate of 4 cm/sec. At what rate are the diagonal of the rectangle and its area increasing when b = 30 cm?

664. The radius of a sphere is increasing at a uniform rate of 5 cm/sec. At what rate increase the. area of the surface and the volume of the sphere when the radius becomes 50 cm?

665. A point is in motion along the spiral of Archimedes$r = a\varphi$ (a = 10 cm) so that the angular velocity of rotation of its radius vector is constant and equals 6° per second. Determine the rate of elongation of the radius vector *r* when *r* = 25 cm.

666. A non-homogeneous bar *ABis* 12 cm long. The mass of a part of it, *AM,* increases with the square of the distance of the moving point *M* from the end *A* and is 10 gm when *AM* =2 cm. Find the mass of the entire .bar *AB* and the linear density at any point *M.* What is the linear density of the bar at *A* and *B*?

<u>Answers 621 – 666</u>

622. 45°; arc tan 2 $\approx$ 63° 26′. **623.** 45°. **624.** arc tan $\frac{2}{e} \approx$ 36° 21′. **625.** (0, 20); (1, 15); (−2, −12). **626.** (1, −3). **627.** $y = x^2 - x + 1$. **628.** $k = \frac{-1}{11}$. **629.** $\left(\frac{1}{8}, -\frac{1}{16}\right)$. **631.** $y - 5 = 0$; $x + 2 = 0$. **632.** $x - 1 = 0$; $y = 0$. **633.** a) $y = 2x$; $y = -\frac{1}{2}x$; b) $x - 2y - 1 = 0$; $2x + y - 2 = 0$; c) $6x + 2y - \pi = 0$; $2x - 6y + 3\pi = 0$; d) $y = x - 1$; $y = 1 - x$; e) $2x + y - 3 = 0$; $x - 2y + 1 = 0$ for the point (1, 1); $2x - y + 3 = 0$; $x + 2y - 1 = 0$ for the point (−1, 1). **634.** $7x - 10y + 6 = 0$, $10x + 7y - 34 = 0$. **635.** $y = 0$; $(\pi + 4)x + (\pi - 4)y - \frac{\pi^2\sqrt{2}}{4} = 0$. **636.** $5x + 6y - 13 = 0$, $6x - 5y + 21 = 0$. **637.** $x + y - 2 = 0$. **638.** At the point (1, 0): $y = 2x - 2$; $y = \frac{1-x}{2}$; at the point (2, 0): $y = -x + 2$; $y = x - 2$; at the point (3, 0): $y = 2x - 6$; $y = \frac{3-x}{2}$. **639.** $14x - 13y + 12 = 0$; $13x + 14y - 41 = 0$.

640. Hint. The equation of the tangent is $\frac{x}{2x_0} + \frac{y}{2y_0} = 1$. Hence, the tangent crosses the x-axis at the point $A(2x_0, 0)$ and the y-axis at $B(0, 2y_0)$. Finding the midpoint of AB, we get the point (x_0, y_0). **643.** 40° 36′. **644.** The parabolas are tangent at the point (0, 0) and intersect at an angle arc tan $\frac{1}{7} \approx$ 8° 8′ at the point (1, 1). **647.** $S_t = S_n = 2$; $t = n = 2\sqrt{2}$. **648.** $\frac{1}{\ln 2}$. **652.** $T = 2a \sin\frac{t}{2} \tan\frac{t}{2}$; $N = 2a \sin\frac{t}{2}$; $S_t = 2a \sin^2\frac{t}{2} \tan\frac{t}{2}$; $S_n = a \sin t$. **653.** arc tan $\frac{1}{k}$. **654.** $\frac{\pi}{2} + 2\varphi$. **655.** $S_t = 4\pi^2 a$; $S_n = a$; $t = 2\pi a\sqrt{1 + 4\pi^2}$; $n = a\sqrt{1 + 4\pi^2}$; $\tan\mu = 2\pi$. **656.** $S_t = a$; $S_n = \frac{a}{\varphi_0^2}$; $t = \sqrt{a^2 + \varrho_0^2}$; $n = \frac{\varrho_0}{a}\sqrt{a^2 + \varrho_0^2}$; $\tan\mu = -\varphi_0$. **657.** 3 cm/sec; 0; −9 cm/sec. **658.** 15 cm/sec. **659.** $-\frac{3}{2}$ m/sec. **660.** The equation of the trajectory is $y = x\tan\alpha - \frac{g}{2v_0^2\cos^2\alpha}x^2$. The range is $\frac{v_0^2 \sin 2\alpha}{g}$. The velocity, $\sqrt{v_2^0 - 2v_0 gt\sin\alpha + g^2t^2}$; the slope of the velocity vector is $\frac{v_0\sin\alpha - gt}{v_0\cos\alpha}$. **Hint.** To determine the trajectory, eliminate the parameter t from the given system. The range is the abscissa of the point A (Fig. 17). The projections of velocity on the axes are $\frac{dx}{dt}$ and $\frac{dy}{dt}$. The magnitude of the velocity is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$; the velocity vector is directed along the tangent to the trajectory. **661.** Diminishes with the velocity 0.4. **662.** $\left(\frac{9}{8}, \frac{9}{2}\right)$. **663.** The diagonal increases at a rate of ~ 3.8 cm/sec, the area, at a rate of 40 cm²/sec. **664.** The surface area increases at a rate of 0.2π m²/sec, the volume, at a rate of 0.05π m³/sec. **665.** $\frac{\pi}{3}$ cm/sec. **666.** The mass of the rod is 360 g, the density at M is $5x$ g/cm, the density at A is 0, the density at B is 60 g/cm.

4.5 L'Hospital - Bernoulli Rule for Evaluating Indeterminate Expressions

4.5.1 Evaluation of the indeterminate forms 0/0 and∞/∞: Let the single-valued functions f(x) and $\varphi(x)$ be differentiable for $0 < |x - a| < h$ and the derivative of one of them not vanish.

If both $f(x)$ and $\varphi(x)$ are infinitesimal or infinite as $x \rightarrow a$, i.e., if the quotient $f(x)/\varphi(x)$ at $x = a$ has one of the indeterminate forms 0/0 or ∞/∞, then

$$\lim_{x\to a} \frac{f(x)}{\varphi(x)} = \lim_{x\to a} \frac{f'(x)}{\varphi'(x)},$$

provided that the limit of the ratio of the derivatives exists.

This rule is also applicable when $a = \infty$.

If the quotient $f'(x)/\varphi'(x)$ yields again at the point $x = a$ an indeterminate form of one of the two above-mentioned types and $f'(x)$ and $\varphi'(x)$ satisfy all the requirements stated above for $f(x)$ and $\varphi(x)$, we can pass to the ratio of second derivatives, etc.

However, note that the limit of the ratio $f(x)/\varphi(x)$ may exist, whereas the ratios of the derivatives do not tend to any limit (Example 809).

4.5.2 Other Indeterminate forms: In order to evaluate an indeterminate form like $0 \cdot \infty$, transform the appropriate product $f_1(x)f_2(x)$, where

$$\lim_{x\to a} f_1(x) = 0 \text{ and } \lim_{x\to a} f_2(x) = \infty,$$

into the quotient

$$\frac{f_1(x)}{\frac{1}{f_2(x)}} \text{ (the form } \frac{0}{0} \text{ (or } \frac{f_2(x)}{\frac{1}{f_1(x)}} \text{ (the form } \frac{\infty}{\infty}\text{)}.$$

In the case of the indeterminate form $\infty - \infty$, one should transform the appropriate difference $f_1(x) - f_2(x)$ into the product

$$f_1(x) \left[1 - \frac{f_2(x)}{f_1(x)}\right]$$

and first evaluate the indeterminate form

$$\frac{f_2(x)}{f_1(x)};$$

if its limit as → is 1, we reduce the expression to

$$\frac{1-\frac{f_2(x)}{f_1(x)}}{\frac{1}{f_1(x)}} \quad \left(\text{the form } \frac{0}{0}\right).$$

The indeterminate forms

$$1^\infty,\ 0^0,\ \infty^0$$

are evaluated by first taking logarithms and then finding the limit of the logarithm of the power

$$[f_1(x)]^{f_2(x)}$$

(which requires evaluating a form like 0·∞.)

In certain cases, it is useful to combine L'Hospital's rule with finding limits by elementary techniques.

Example 1 . Compute

$$\lim_{x\to 0} \frac{\ln x}{\cot x} \quad \left(\text{form } \frac{\infty}{\infty}\right).$$

Solution: Applying L'Hospital's rule, we have

$$\lim_{x\to 0} \frac{\ln x}{\cot x} = \lim_{x\to 0} \frac{(\ln x)'}{(\cot x)'} = -\lim_{x\to 0} \frac{\sin^2 x}{x}.$$

We have the indeterminate form 0/0. However, we do not need to use L'Hospital's rule, since we know that

$$\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \sin x = 1 \cdot 0 = 0.$$

Thus, finally, we find

$$\lim_{x \to 0} \frac{\ln x}{\cot x} = 0.$$

Example2. Compute

$$\lim_{x \to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right) \text{ (form } \infty - \infty\text{).}$$

Reducing this to a common denominator, we get

$$\lim_{x \to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right) = \lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} \text{ (form } \frac{0}{0}\text{).}$$

Before applying L'Hospital's rule, we replace the denominator of the last fraction by an equivalent infinitesimal (1.4)$x^2 \sin^2 x \sim x^4$. Thus, we obtain

$$\lim_{x \to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right) = \lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^4} \text{ (form } \frac{0}{0}\text{).}$$

L'Hospital's rule yields now

$$\lim_{x \to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right) = \lim_{x \to 0} \frac{2x - \sin 2x}{4x^3} = \lim_{x \to 0} \frac{2 - 2\cos 2x}{12x^2}.$$

We find now by elementary means

$$\lim_{x \to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right) = \lim_{x \to 0} \frac{1 - \cos 2x}{6x^2} = \lim_{x \to 0} \frac{2\sin^2 x}{6x^2} = \frac{1}{3}.$$

Example 3: Compute

$$\lim_{x \to 0} (\cos 2x)^{\frac{3}{x^2}} \text{ (form } 1^{\infty}\text{).}$$

Taking logarithms and applying L'Hospital's rule, we get

$$\lim_{x \to 0} \ln (\cos 2x)^{\frac{3}{x^2}} = \lim_{x \to 0} \frac{3 \ln \cos 2x}{x^2} = -6 \lim_{x \to 0} \frac{\tan 2x}{2x} = -6,$$

whence

$$\lim_{x\to 0}(\cos 2x)^{\frac{3}{x^2}} = e^{-6}.$$

Exercises 776 – 808 . Find the liimits

776. $\lim\limits_{x\to 1}\dfrac{x^3-2x^2-x+2}{x^2-7x+6}.$

Solution:

$$\lim_{x\to 1}\frac{x^3-2x^2-x+2}{x^3-7x+6} = \lim_{x\to 1}\frac{3x^2-4x-1}{3x^2-7} = \frac{1}{2}\ .$$

777. $\lim\limits_{x\to 0}\dfrac{x cosx-\sin x}{x^3}.$ **778**. $\lim\limits_{x\to 1}\dfrac{1-x}{1-\sin\frac{\pi x}{2}}$ **780**. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x-\sin x}$ **782**. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\tan x}{\tan 5x}$

783. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^5}$. **784**. $\lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt[3]{x}}$. **785**. $\lim\limits_{x\to 0}\dfrac{\frac{\pi}{x}}{\cot\frac{\pi x}{2}}$. **786**. $\lim\limits_{x\to 0}\dfrac{\ln(\sin mx)}{\ln\sin x}$. **787**. $\lim\limits_{x\to 0}(1-\cos x)\cot x$.

Solution:

$$\lim_{x\to 0}(1-\cos x)\cot x = \lim_{x\to 0}\frac{(1-\cos x)\cos x}{\sin x} = \lim_{x\to 0}\frac{(1-\cos x)\cdot 1}{\sin x} = \lim_{x\to 0}\frac{\sin x}{\cos x} = 0.$$

788. $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}$. **789**. $\lim\limits_{x\to 0}\arcsin x cot\ x$. **790**. $\lim\limits_{x\to 0}(x^n e^{-x}), n>0$. **791**. $\lim\limits_{x\to\infty} x sin\ \dfrac{a}{x}$.

792. $\lim\limits_{x\to\infty} x^n sin\ \dfrac{a}{x}, n>0$. **793**. $\lim\limits_{x\to 1}\ln x ln\ (x-1)$.

794. $\lim\limits_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$.

Solution. $\lim\limits_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right) = \lim\limits_{x\to 1}\dfrac{x\ln x - x + 1}{(x-1)\ln x} =$

$$= \lim_{x\to 1}\frac{x\cdot\frac{1}{x}+\ln x-1}{\ln x+\frac{1}{x}(x-1)} = \lim_{x\to 1}\frac{\ln x}{\ln x-\frac{1}{x}+1} = \lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{1}{2}\ .$$

795. $\lim\limits_{x\to 3}\left(\dfrac{1}{x-3}-\dfrac{5}{x^2-x-6}\right)$. **796**. $\lim\limits_{x\to 1}\left[\dfrac{1}{2(1-\sqrt{x})}-\dfrac{1}{3(1-\sqrt[3]{x})}\right]$. **797**. $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{x}{\cot x}-\dfrac{\pi}{2\cos x}\right)$.

798. $\lim\limits_{x\to 0} x^x$.

Solution. We have $x^x = y$; $\ln y = x\ln x$; $\lim\limits_{x\to 0}\ln y = \lim\limits_{x\to 0} x\ln x =$

$$= \lim_{x\to 0}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0,$$ whence $\lim\limits_{x\to 0} y = 1$, that is, $\lim\limits_{x\to 0} x^x = 1$.

799. $\lim\limits_{x\to+\infty} x^{\frac{1}{x}}$ **800**. $\lim\limits_{x\to 0} x^{\frac{3}{4+\ln x}}$. **801**. $\lim\limits_{x\to 0} x^{\sin x}$. **802**. $\lim\limits_{x\to 1}(1-x)^{\cos\frac{\pi x}{2}}$. **803**. $\lim\limits_{x\to}(1+x^2)^{\frac{1}{x}}$.

804. $\lim\limits_{x\to 1} x^{\frac{1}{1-x}}$. **805**. $\lim\limits_{x\to 1}(\tan\frac{\pi x}{4})^{\tan\frac{\pi x}{4}}$. **806**. $\lim\limits_{x\to 0}\cot x^{\frac{1}{\ln x}}$ **807**. $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$. **808**. $\lim\limits_{x\to 0}\cot x^{\sin x}$.

Answers 777 - 808

777. $-\frac{1}{3}$ .778. ∞. 779. 1. 780. 3. 781. $\frac{1}{2}$. 782. 5. 783. ∞. 784. 0. 785. $\frac{\pi^2}{2}$.
786. 1. 788. $\frac{2}{\pi}$. 789. 1. 790. 0. 791. a. 792. ∞ for $n>1$; a for $n=1$;
0 for $n<1$. 793. 0. 795. $\frac{1}{5}$. 796. $\frac{1}{12}$. 797. -1. 799. 1. 800. e^3. 801. 1.
802. 1. 803. 1. 804. $\frac{1}{e}$. 805. $\frac{1}{e}$. 806. $\frac{1}{e}$. 807. 1. 808. 1.

4.6. Extreme values of a Function of One Argument

4.6.1 Increase and decrease of functions: The function f = f(x)) is said to be increasing (decreasing) in some interval if, for any points x_1 and x_2 which belong to this interval, the inequality $x_1 < x_2$ yields the inequality $f(x_1) < f(x_2)$ (Fig. 2l a) . [$f(x_1) > f(x_2)$)Fig. 21b)] If f(x) is continuous in the interval [a, b] and $f'(x) > 0$ [$f'(x) < 0$] for a < x < b, then f(x) increases (decreases) in the interval [a, b].

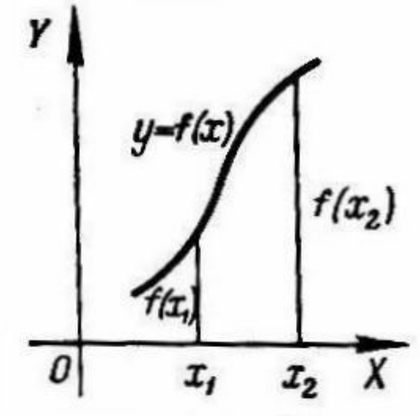

Fig.21 (a)

In the simplest cases, the domain of definitiom of *f*(*x*) may be subdivided into a finite number of intervals of increase and decrease of the function (intervals of monotone behaviour). These intervals are bounded by critical points *x* [where *f'*(*x*)=0 or *f'*(*x*) does not exist].

Example 1. Test for increase and decrease: the function:

$$y' = 2x - 2 = 2(x-1).$$

Solution: We find the derivative$y' = 2x - 2 = 2(x-1)$, whence $y' = 0$ for $x = 1$. On a number scale, we get the intervals of monotone behavior: (1) (-∞, 1), (2) (1, +∞). From (1), we have:

a) if -∞< x <l, then y ' > 0, whence the function *f*(*x*) decreases In the interval (-∞, 1);
b) if 1 < x < +∞, then y ' > 0, whence the function *f*(*x*) increases in the interval (1, +∞) (Fig, 22).

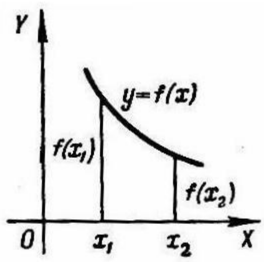

Fig.21 (b)

Example2. Determine the intervals of increase and decrease of the function$y = \frac{1}{x+2}$.
Solution: Here, *x* = -2 is a discontinuity of the function and *y'*=-1/(x+2)²<2 for x^1-2, whence the function *y* decreases in the intervals ¥ < x < -2 and -2 < x < +∞.

Example3. Test for increase or decrease the function $y = \frac{1}{5}x^5 - \frac{1}{3}x^3$.

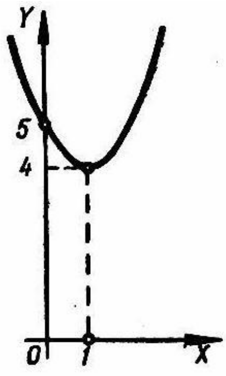

Fig. 22

Solution: Here,$y' = x^4 - x^2$. (2)

Solving the equation $x^4 - x^2 = 0$, we find the points x_1=-1, x_2=0, x_3=l, at which the derivative *y'* vanishes. Since *y'* can change sign only when passing through points at which it vanishes or becomes discontinuous (in the given case, *y'* has no discontinuities!), the derivative in each of the intervals (-∞, -1), (-1, 0), (0,1) and (1, +∞) retains its sign, whence the function is monotonic in each of these intervals. In order to determine in which of these intervals the function increases and decreases, one has to determine the sign of the derivative in each interval. In order to determine what the sign of *y'* is in the interval (-∞,-1), it is sufficient to determine the sign of *y'* at some point of the interval, for example, we find from (2), for x = -2, y'=12 > 0, whence *y'* >0 in the interval (-∞, -1) and the function increases in this interval. Similarly, we find that *y'* <0 in the interval (-1,0) (as a check, we can take x = -1/2), y' < 0 in the interval (0,1) (here we can use x = 1/2) and y' > 0 in the interval (1, +∞).

Thus, the function being tested increases in the interval (-∞, -1), decreases in the interval (-1, 1) and again increases in the interval (1, +∞).

<u>4.6.2 Extreme values of a function:</u> If there exists a two-sided neighbourhood of a point x_0 such that for every point $x^1 x_0$ of this neighborhood we have the inequality f(x) > $f(x_0)$, then the point *x* is called a minimum point of the function *y = f(x)*, while the number $f(x_0)$ is called the minimum of the function *y = f(x)*. Similarly, if for any point $x^1 x_1$ of some neighborhood of the point x_1 the inequality $f(x) < f(x_1)$ *is* fulfilled, then x_1 is called the maximum point of the function *f*(*x*) and $f(x_1)$ *is* the maximumof the function (Fig. 23).

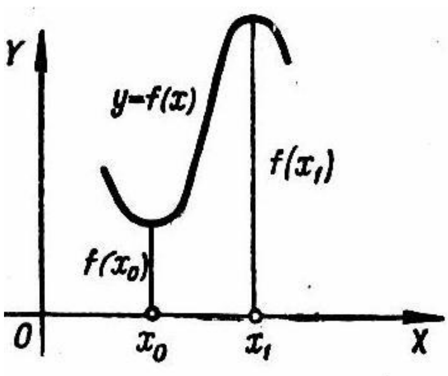

Fig. 23

The minimum point or maximum point of a function is its extreme point and the minimum or maximum of a function is called the extremum value of the function. If x_0 is an extreme point of the function *f(x),* then $f'(x_0)$=0 or $f'(x_0)$ does not exist (a necessary condition for the existence of an extreme value). The converse is not true: Points at which *f'*(*x*) = 0 or *f'*(*x*)does not exist (critical points) are not necessarily extreme points of the function *f(x).* The sufficient conditions for the existence and absence of an extreme value of a continuous function f(*x*) are given by the rules:

1. If there exists a neighborhood (x_0 - d, x_0 + d) of a critical point x_0 such that f'(x) > 0 for x_0-d<x<x_0 and *f'*(*x*)<0 for x_0<x<x_0+d, then x_0 is the maximum point of the function *f*(*x*), and if *f'*(*x*)<0 for x_0-d<x<x_0 and *f* '(*x*)>0 for x_0<x<x_0+d, then x_0 is a minimum point of the function *f*(*x*).

Finally, if there is some positive number d such that *f'*(*x*) retains its sign unchanged for 0</x-x_0/<d€ž then x_0 is not an extreme point of the function f(x).

2. If *f* '(x) = 0 and *f* "(x_0)<0, then x_0 is a maximum point; if *f* '(x_0)=0 and *f* "(x_0) >0, then x_0 is a minimum point; however, if *f'*(x_0) = 0,*f* "(x_0) = 0and *f* ''' (x_0) ^{1} 0, then the point x_0 is not an extreme point.

More generally, let the first of the non-zero derivatives (not equal to zero at the point x_0) of the function *f*(*x*) be of the order *k.* Then, if *k* is even, the point x_0 is an extreme point, namely, a maximum point, if $f^{(k)}(x_0)$ <0, and a minimum point, if $f^{(k)}(x_0)$ >0. However, if *k* is odd, then x_0 is not an extreme point.

Example 4. Find the extreme values of the function $y = 2x + 3\sqrt[3]{x^2}$.
Solution: The first derivative is

$$y' = 2 + \frac{2}{\sqrt[3]{x}} = \frac{2}{\sqrt[3]{x}}(\sqrt[3]{x} + 1). \qquad (3)$$

Setting the derivative *y'* equal to zero, we get: $\sqrt[3]{x} + 1 = 0$, whence we find the critical point x_1 = -1. By (3), we have: If *x* = -1 - *h,* where *h* is a sufficiently small positive number, then *y'* >0;on the other hand, if *x* = -1 + *h,* then *y'* <0*, whence, x_1 = -1 is maximum point of the function *y* and y_{max} = l.

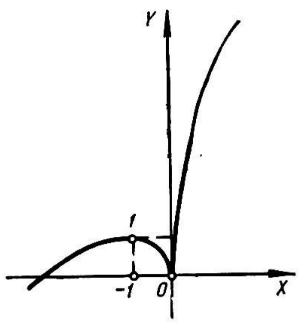

Fig. 24

* If it is difficult to determine the sign of the derivative *y',* one can calculate arithmetically by taking for *h* a sufficiently small positive number.

Equating the denominator of the expression for *y'* in (3) to zero, we get $\sqrt[3]{x} = 0$, whence we find the second critical point of the function x_2 = 0, where there is no derivative. For *x* = -*h,* we have obviously *y'* <0, for *x*=*h,* we have y ' > 0. Consequently, x = 0 is the minimum point of the function *y* and y_{min} = 0 (Fig. 24). It is also possible to test the behaviour of the function at the point *x* = -1 by means of the second derivative $y'' = -\frac{2}{3x\sqrt[3]{x}}$.

Here, *y* " <0 for x = -1, whence x = -1 is the maximum point of the function.

<u>4.6.3 Largest and smallest values:</u> The smallest (largest) value of a continuous function *f*(*x*) in a given interval [a, b] is attained either at the critical points of the function or at the end points of the interval [a, b].

Example 5. Find the largest and smallest values of the function $y = x^3 - 3x + 3$ on the interval $-1½ \le x \le 2½$.

Solution: Since $y' = 3x^2 - 3$, it follows that the critical points of the function *y* are x_1= -1 and x_2 = 1. Comparing the values of .the function at

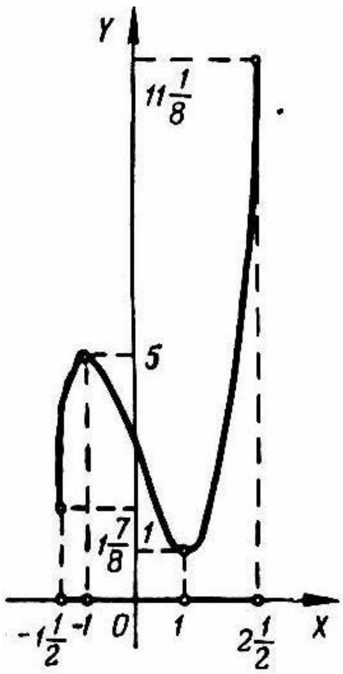

Fig. 25

these points and the values of the function at the end-points of the given intervall

$$y(-1) = 5; \quad y(1) = 1; \quad y(-1{,}5) = 4\frac{1}{8};$$

we conclude (Fig. 25) that the function attains its smallest value, m = l, at the point x = 1 (the minimum point) and its largest value *M*= 11 1/8 at the point x = 2 1/8, (at the right hand end point of the interval).

<u>Exercises 811 - 854</u> Determine the intervals of decrease and increase of the functions:

811. $y = 1 - 4x - x^2$.

812. $y = (x-2)^2$.

813. $y = (x+4)^3$.

814. $y = x^2(x-3)$.

815. $y = \frac{x}{x-2}$.

816. $y = \frac{1}{(x-1)^2}$.

817. $y = \frac{x}{x^2 - 6x - 16}$.

818. $y = (x-3)\sqrt{x}$.

819. $y = \frac{x}{3} - \sqrt[3]{x}$.

820. $y = x + \sin x$.

821. $y = x \ln x$.

822. $y = \text{arsin}\,(1+x)$.

823. $y = 2e^{x^2-4x}$.

824. $y = 2^{\frac{1}{x-a}}$.

825. $y = \frac{e^x}{x}$.

826. Test the functions for extreme values: $y = x^2 + 4x + 6$.

Solution: We find the derivative y' = 2x + 4. Setting y' = 0, we obtain the critical value x = -2. Since y' < 0 when x < -2, it follows x = -2 is the minimum and $y_{min} = 2$. We get the same result by using the sign of the second derivative at the critical point y'' => 0.

827. $y = 2 + x - x^2$.

828. $y = x^3 - 3x^2 + 3x + 2$.

829. $y = 2x^3 + 3x^2 - 12x + 5$.

Solution: We find the derivative $y' = 6x^2 + 6x - 12 = 6\,(x^2 + x - 2)$.

Equating the derivative y' to zero, we find the critical points $x_1 = -2$ and $x_2 = l$. In order to determine the nature of the extreme value, we calculate the second derivative y''=6(2x + l). Since *y*'' (-2) < 0, it follows that $x_1 = -2$ is the maximum point of the function *y* and $y_{max} = 25$. Similarly, we have y''(1) > 0, whence $x_2 = l$ is the minimum point of the function *y* and $y_{min} = -2$.

830. $y=x^2(x-12)^2$.

831. $y=x(x-1)^2(x-2)^3$.

832. $y=\frac{x^3}{x^2+3}$.

833. $y=\frac{x^2-2x+2}{x-1}$.

834. $y=\frac{(x-2)(8-x)}{x^2}$.

835. $y=\frac{16}{x(4-x^2)}$.

836. $y=\frac{4}{\sqrt{x^2+8}}$.

837. $y=\frac{x}{\sqrt[3]{x^2-4}}$.

838. $y=\sqrt[3]{(x^2-1)^2}$.

839. $y=2\sin 2x+\sin 4x$.

840. $y=2\cos\frac{x}{2}+3\cos\frac{x}{3}$.

841. $y=x-\ln(1+x)$.

842. $y=x\ln x$.

843. $y=x\ln^2 x$.

844. $y=\cosh x$.

845. $y=xe^x$.

846. $y=x^2e^{-x}$.

847. $y=\frac{e^x}{x}$.

848. $y=x-\text{artan }x$.

Determine the smallest and largest values of the functions in the indicated intervals (if the interval is not given, determine the smallest and largest values of the function throughout the domain of definition.

849. $y=\frac{x}{1+x^2}$.

850. $y=\sqrt{x(10-x)}$.

851. $y=\sin^4 x+\cos^4 x$.

852. $y=\arccos x$.

853. $y=x^3$ on the interval $[-1,3]$.

854. $y=2x^3+3x^2-12x+1$

a) on the interval $[-1,5]$;

b) on the interval $[-10,12]$.

Answers 811 – 854

811. $(-\infty, -2)$, increases; $(-2, \infty)$, decreases. **812.** $(-\infty, 2)$, decreases; $(2, \infty)$, increases. **813.** $(-\infty, \infty)$, increases. **814.** $(-\infty, 0)$ and $(2, \infty)$, increases; $(0, 2)$, decreases. **815.** $(-\infty, 2)$ and $(2, \infty)$, decreases. **816.** $(-\infty, 1)$, increases; $(1, \infty)$, decreases. **817.** $(-\infty, -2)$, $(-2, 8)$ and $(8, \infty)$, decreases. **818.** $(0, 1)$, decreases; $(1, \infty)$, increases. **819.** $(-\infty, -1)$ and $(1, \infty)$, increases; $(-1, 1)$, decreases. **820.** $(-\infty, \infty)$, increases. **821.** $\left(0, \frac{1}{e}\right)$, decreases; $\left(\frac{1}{e}, \infty\right)$, increases. **822.** $(-2, 0)$, increases. **823.** $(-\infty, 2)$, decreases; $(2, \infty)$, increases. **824.** $(-\infty, a)$ and (a, ∞), decreases. **825.** $(-\infty, 0)$ and $(0, 1)$, decreases; $(1, \infty)$, increases. **827.** $y_{\max}=\frac{9}{4}$ when $x=\frac{1}{2}$. **828.** No

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extremum. **830.** $y_{min}=0$ when $x=0$; $y_{min}=0$ when $x=12$; $y_{max}=1296$ when $x=6$. **831.** $y_{min}\approx -0.76$ when $x\approx 0.23$; $y_{max}=0$ when $x=1$; $y_{min}\approx -0.05$ when $x\approx 1.43$. No extremum when $x=2$. **832.** No extremum. **833.** $y_{max}=-2$ when $x=0$; $y_{min}=2$ when $x=2$. **834.** $y_{max}=\frac{9}{16}$ when $x=3.2$. **835.** $y_{max}=-3\sqrt{3}$ when $x=-\frac{2}{\sqrt{3}}$; $y_{min}=3\sqrt{3}$ when $x=\frac{2}{\sqrt{3}}$. **836.** $y_{max}=\sqrt{2}$ when $x=0$. **837.** $y_{max}=-\sqrt{3}$ when $x=-2\sqrt{3}$; $y_{min}=\sqrt{3}$ when $x=2\sqrt{3}$. **838.** $y_{min}=0$ when $x=\pm 1$; $y_{max}=1$ when $x=0$. **839.** $y_{min}=-\frac{3}{2}\sqrt{3}$ when $x=\left(k-\frac{1}{6}\right)\pi$; $y_{max}=\frac{3}{2}\sqrt{3}$ when $x=\left(k+\frac{1}{6}\pi\right)$ $(k=0, \pm 1, \pm 2, \ldots)$. **840.** $y_{max}=5$ when $x=12k\pi$; $y_{max}=5\cos\frac{2\pi}{5}$ when $x=12\left(k\pm\frac{2}{5}\right)\pi$; $y_{min}=-5\cos\frac{\pi}{5}$ when $x=12\left(k\pm\frac{1}{5}\right)\pi$; $y_{min}=1$ when $x=6(2k+1)\pi$ $(k=0,$

$\pm 1, \pm 2, \ldots)$. **841.** $y_{min}=0$ when $x=0$. **842.** $y_{min}=-\frac{1}{e}$ when $x=\frac{1}{e}$. **843.** $y_{max}=\frac{4}{e^2}$ when $x=\frac{1}{e^2}$; $y_{min}=0$ when $x=1$. **844.** $y_{min}=1$ when $x=0$. **845.** $y_{min}=-\frac{1}{e}$ when $x=-1$. **846.** $y_{min}=0$ when $x=0$; $y_{max}=\frac{4}{e^2}$ when $x=2$. **847.** $y_{min}=e$ when $x=1$. **848.** No extremum. **849.** Smallest value is $m=-\frac{1}{2}$ for $x=-1$; greatest value, $M=\frac{1}{2}$ when $x=1$. **850.** $m=0$ when $x=0$ and $x=10$; $M=5$ for $x=5$. **851.** $m=\frac{1}{2}$ when $x=(2k+1)\frac{\pi}{4}$; $M=1$ for $x=\frac{k\pi}{2}$ $(k=0, \pm 1, \pm 2, \ldots)$. **852.** $m=0$ when $x=1$; $M=\pi$ when $x=-1$. **853.** $m=-1$ when $x=-1$; $M=27$ when $x=3$. **854.** a) $m=-6$ when $x=1$; $M=266$ when $x=5$; b) $m=-1579$ when $x=-10$; $M=3745$ when $x=12$.

4.7. The Direction of Concavity. Points of Inflection

4.7.1 The concavity of the graph of a function: We say that the graph of a differentiable function $y = f(x)$ is concave downwards in the interval (a,b) (concave upwards in the interval (a_1, b_1), it for $a < x < b$ the arc of the curve is below (or for $a_1<x<b_1$ above) the tangent drawn at any point of the interval (a,b) or of the interval (a_1, b_1)] (Fig. 29). A sufficient condition for downwards (upwards) concave behaviour of a graph $y = f(x)$ is that there is fulfilled in the appropriate interval the inequality:

$$f''(x) < 0 \quad [f''(x) > 0].$$

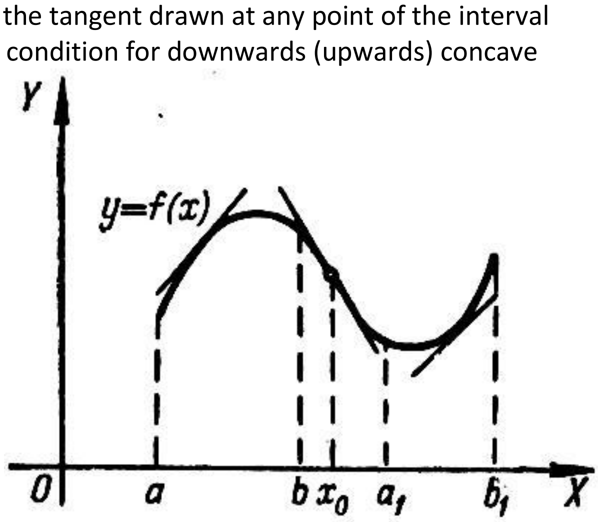

Fig. 29

4.7.2 Points of Inflection: A point $[x_0, f(x_0)]$ at which the direction of concavity of the graph of some function changes is called a point of inflection (Fig. 29).

For the abscissa of the point of inflection x_0 of the graph of a function $y = f(x)$, there is no second derivative $f''(x_0) = 0$ or $f''(x_0)$. Points at which $f''(x)=0$ or $f''(x)$ does not exist are called critical points of the second kind. The critical point of the second kind x_0 is the abscissa of the point of inflection if $f''(x)$ retains constant signs in the intervals $x_0-\delta<x<x_0$ and $x_0<x<x_0+\delta$, where $\delta$is some positive number; provided these signs are opposite. And it is not a point of inflection, if the signs of $f''(x)$ are the same in the intervals indicated above.

Example1. Determine the intervals of concavity and convexity and also the points of inflection of the Gaussian curve$y = e^{-x^2}$.

Solution: We have$y' = -2xe^{-x^2}$, $y'' = (4x^2 - 2)e^{-x^2}$. Setting the second derivative equal to zero, we find the critical points of the second kind

$$x_1 = -\frac{1}{\sqrt{2}}, \quad x_2 = \frac{1}{\sqrt{2}}.$$

These points subdivide the number scale $-\infty < \xi < \infty$ into the three intervals:

$$\text{I } (-\infty, x_1), \text{ II } (x_1, x_2), \text{ III } (x_2, +\infty).$$

The signs of y'' are +, - , + respectively (this is obvious, if, for example, we take one point in each interval and substitute the corresponding values of x into y). Hence,

(1) the curve is concave upwards when $-\infty< x < -1/2$ and $1/2 < x < +\infty$
(2) the curve is concave downwards when $-1/2 < x <+1/2$.

The points (±1/2, 1/-1/e) are points of inflection. (Fig. 30).

Note that due to the symmetry about the y-axis of the Gaussian curve curve, it would have been sufficient to investigate the sign of the concave behaviour of this curve on the semi-axis $0 < x < +\infty$.

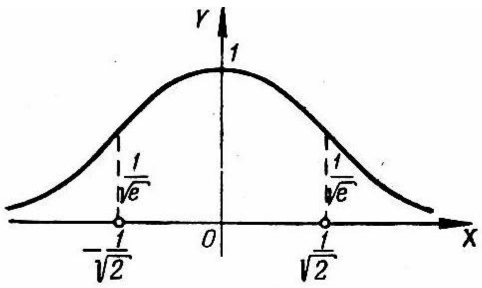

Fig. 30

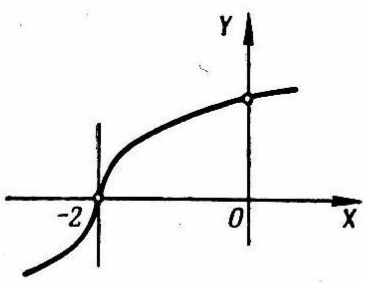

Fig. 31

Example 2. Find the points of inflection of the graph of the function$y = \sqrt[3]{x+2}$.
Solution: We have:$y'' = \frac{-2}{9\sqrt[3]{(x+2)^5}}$. (1)

Obviously, y" does not vanish anywhere. Equating to zero the denominator of the fraction on the right hand side of (1), we find that y" does not exist for x < -2. Since y" > 0 for $x < -2$ and y" < 0 for x >-

2, it follows that (-2,0) is the point of inflection (Fig. 31). The tangent at this point is parallel to the ordinate axis, since the first derivative *y'* is infinite at x = -2.

Exercises 891 - 900

Find the intervals of concavity and the points of inflection of the graphs of the functions:

891. $y = x^3 - 6x^2 + 12x + 4$.

892. $y = (x+1)^4$.

893. $y = \frac{1}{x+3}$.

894. $y = \frac{x^3}{x^2+12}$.

895. $y = \sqrt[3]{4x^3 - 12x}$.

896. $y = \cos x$.

897. $y = x - \sin x$.

898. $y = x^2 \ln x$.

899. $y = \arctan x - x$.

900. $y = (1 + x^2) e^x$.

Answers 891 – 900

891. $(-\infty, 2)$, concave down; $(2, \infty)$, concave up; $M(2, 12)$, point of inflection. 892. $(-\infty, \infty)$, concave up. 893. $(-\infty, -3)$, concave down, $(-3, \infty)$, concave up; no points of inflection. 894. $(-\infty, -6)$ and $(0, 6)$, concave up; $(-6, 0)$ and $(6, \infty)$, concave down; points of inflection $M_1\left(-6, -\frac{9}{2}\right)$, $O(0, 0)$, $M_2\left(6, \frac{9}{2}\right)$. 895. $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$, concave up; $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$, concave down; points of inflection $M_{1,2}(\pm\sqrt{3}, 0)$ and $O(0, 0)$. 896. $\left((4k+1)\frac{\pi}{2}, (4k+3)\frac{\pi}{2}\right)$, concave up; $\left((4k+3)\frac{\pi}{2}, (4k+5)\frac{\pi}{2}\right)$, concave down $(k = 0, \pm 1, \pm 2, \ldots)$; points of inflection, $\left((2k+1)\frac{\pi}{2}, 0\right)$. 897. $(2k\pi, (2k+1)\pi)$, concave up; $((2k-1)\pi, 2k\pi)$, concave down $(k = 0, \pm 1, \pm 2, \ldots)$; the abscissas of the points of inflection are equal to $x = k\pi$. 898. $\left(0, \frac{1}{\sqrt{e^3}}\right)$, concave down; $\left(\frac{1}{\sqrt{e^3}}, \infty\right)$, concave up; $M\left(\frac{1}{\sqrt{e^3}}, -\frac{3}{2e^3}\right)$ is a point of inflection. 899. $(-\infty, 0)$, concave up; $(0, \infty)$, concave down; $O(0, 0)$ is a point of inflection. 900. $(-\infty, -3)$ and $(-1, \infty)$, concave up; $(-3, -1)$, concave down; points of inflection are $M_1\left(-3, \frac{10}{e^3}\right)$ and $M_2\left(-1, \frac{2}{e}\right)$.

4.8. Asymptotes

4.8.1. Definition: If a point (*x,y*) is in continuous motion along a curve y = *f*(x) in such a way that at least one of its co-ordinates approaches infinity (and at the same time. the distance of the point from some straight line tends to zero), then this straight line is called an asymptote of the curve.

4.8.2 Vertical asymptotes: If there is a number *a* such that a such that

$$\lim_{x \to a} f(x) = \pm\infty,$$

then the straight line *x* = *a* is avertical asymptote.

4.8.3 Inclined asymptotes: If there are liimits

$$\lim_{x\to\infty}\frac{f(x)}{x}=k_1,\quad \lim_{x\to\infty}[f(x)-k_1x]=b_1.$$

then the straight line $y = k_1x + b_1$ will be an asymptote (a right inclined asymptote or, when $k_1 = 0$, a right horizontal asymptote).

If there are liimits

$$\lim_{x\to-\infty}\frac{f(x)}{x}=k_2,\quad \lim_{x\to-\infty}[f(x)-k_2x]=b_2.$$

then the straight line $y = k_2x + b_2$ is an asymptote (a left inclined asymptote or, when $k_2 = 0$, a left horizontal asymptote). The graph of the function $y = f(x)$ (we assume the function is single-valued) cannot have more than one right (inclined or horizontal) and more than one left (inclined or horizontal) asymptote.

Example 1. Find the asymptotes of the curve $y=\frac{x^2}{\sqrt{x^2-1}}$

Solution: Equating the denominator to zero, we get two vertical asymptotes: $x=-1,\ x=1$.

We seek the inclined asymptotes. For $x\to\infty$, we obtain

$$k_1=\lim_{x\to+\infty}\frac{y}{x}=\lim_{x\to+\infty}\frac{x^2}{x\sqrt{x^2-1}}=1,$$

$$b_1=\lim_{x\to+\infty}(y-x)=\lim_{x\to+\infty}\frac{x^2-x\sqrt{x^2-1}}{\sqrt{x^2-1}}=0,$$

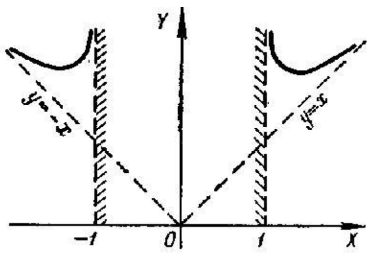

Fig. 32

whence, the straight line $y = x$ is the right asymptote. Similarly, when $x\to-\infty$, we have

$$k_2=\lim_{x\to-\infty}\frac{y}{x}=-1,$$

$$b_2=\lim_{x\to-\infty}(y+x)=0.$$

Thus, the left asymptote is $y = -x$ (Fig. 32). Testing a curve for asymptotes is simplified if we take into consideration a curve's symmetry.

Example 2: Find the asymptotes of the curve $y=x+\ln x$.
Solution: Since

$$\lim_{x\to 0+,} y=-\infty,$$

the straight line $x = 0$ is a vertical asymptote (lower). Let us now test the curve only for the inclined right asymptote (since $x>0$).

We have

$$k = \lim_{x\to\infty,} \frac{y}{x} = 1, \quad b = \lim_{x\to\infty,} (y - x) = \lim_{x\to\infty,} \ln x = \infty.$$

Hence, there is no inclined asymptote.

If a curve is represented by the parametric equations $x = \varphi(t)$, $y = \psi(t)$, we first test to find out whether there are any values of the parameter t for which one of the functions $\varphi(t)$ or $\psi(t)$ becomes infinite, while the other remains finite. When $\varphi(t_0) = \infty$ and $\psi(t_0) = c$, the curve has a horizontal asymptote $y = c$. When $\psi(t_0) = \infty$ and $\varphi(t_0) = c$, the curve has a vertical asymptote $x = c$.

If

$$\varphi(t_0) = \psi(t_0) = \infty, \quad \lim_{t\to t_0} \frac{\psi(t)}{\varphi(t)} = k, \quad \lim_{t\to t_0} [\psi(t) - k\varphi(t)] = b,$$

the curve has an inclined asymptote $y = kx + b$.

If a curve is represented by a polar equation $r = f(\varphi)$, we can find its asymptotes by the preceding rule after transforming the equation of the curve to the parametric form by the formulae

$$x = r\cos\varphi = f(\varphi)\cos\varphi, \quad y = r\sin\varphi = f(\varphi)\sin\varphi.$$

Exercises 901 - 915

Find the asymptotes of the curves:

901. $y = \frac{1}{(x-2)^2}$.

902. $y = \frac{x}{x^2 - 4x + 3}$.

903. $y = \frac{x^2}{x^2 - 4}$.

904. $y = \frac{x^3}{x^2 + 9}$.

905. $y = \sqrt{x^2 - 1}$.

906. $y = \frac{x}{\sqrt{x^2 + 3}}$.

907. $y = \frac{x^2 + 1}{\sqrt{x^2 - 1}}$.

908. $y = x - 2 + \frac{x^2}{\sqrt{x^2 + 9}}$.

909. $y = e^{-x^2} + 2$.

910. $y = \frac{1}{1 - e^x}$.

911. $y = e^{\frac{1}{x}}$.

912. $y = \frac{\sin x}{x}$.

913. $y = \ln(1 + x)$.

914. $x = t$; $y = t + 2 \operatorname{artan} t$.

915. Find the asymptote of the hyperbolic spiral $r = \frac{a}{\varphi}$.

Answers 905– 914

905. $y = -x$, left, $y = x$, right. **906.** $y = -1$, left, $y = 1$, right. **907.** $x = \pm 1$, $y = -x$, left, $y = x$, right. **908.** $y = -2$, left, $y = 2x - 2$, right. **909.** $y = 2$. **910.** $x = 0$, $y = 1$, left, $y = 0$, right. **911.** $x = 0$, $y = 1$. **912.** $y = 0$. **913.** $x = -1$. **914.** $y = x - \pi$, left; $y = x + \pi$, right.

4.9. Graphing Functions by Characteristic Points
When constructing the graph of a function, first find its domain of definition and then determine the behavior of the function on the boundary of this domain. It is also useful to note any peculiarities of the function (if there are any) such as symmetry, periodicity, constancy of sign, monotonic behavior, etc. Then, find any points of discontinuity, bending points, points of inflection, asymptotes, etc. These elements help to determine the general nature of the graph of the function and to obtain a mathematically correct outline of it.

Example 1. Construct the graph of the function$y = \frac{x}{\sqrt[3]{x^2-1}}$

Solution: a) The function exists everywhere except at the points *x* = ±l. It is odd, whence the graph is symmetric about the point O(0, 0). This simplifies the construction of the graph.

b) Its discontinuities are *x* = -1 and *x* = 1*,* and

$$\lim_{x\to 1+0} y = \pm\infty, \quad \lim_{x\to -1\pm 0} y = \pm\infty,$$

whence the straight lines *x*= ±l are vertical asymptotes.

c) We seek inclined asymptotes and find

$$k_1 = \lim_{x\to +\infty} \frac{y}{x} = 0, \quad b_1 = \lim_{x\to +\infty} y = \infty,$$

whence there is no right asymptote. It follows from the symmetry of the curve that there is also no left-hand asymptote.

d) We find critical points of the first and second kinds, i.e., points at which the first or the second derivative, respectively, vanishes or does not exist.
We have

$$y' = \frac{x^2-3}{3\sqrt[3]{(x^2-1)^4}}, \qquad y'' = \frac{2x(9-x^2)}{9\sqrt{(x^2-1)^7}}$$

The derivatives y and y" only do not exist at *x* = ±1*,* i.e., only at points where the function *y* itself does not exist, whence the critical points are only those at which *y'* and *y"* vanish.

It follows from he derivatives y and y"that

$$y' = 0, when\ x = \pm\sqrt{3}, \quad y'' = 0, when\ x = 0\ and\ x = \pm 3.$$

Thus, *y'* retains a constant sign in each of the intervals

$$(-\infty; .-\sqrt{3}), \quad (-\sqrt{3}; -1), \quad (-1; 1), \quad (1; \sqrt{3}), \quad (-\sqrt{3}; \infty)$$

and *y"* in each of the intervals

$$(-\infty; -3), \quad (-3; -1), \quad (-1; 0), \quad (0; 1), \quad (1; 3), \quad (3; \infty).$$

In order to determine the signs of *y'* (or *y"*, respectively) in each of these intervals, it is sufficient te determine the sign of *y'* (or *y"*) at any point of each of these intervals.

It is convenient to make a table of the results of such an investigation (Table I*)*, calculating also the ordinates of the characteristic points of the graph of the function. It will be noted that due to the odd character of the function *y,* it is enough to calculate only for $x \geq 0$; the left half of the graph is constructed by the principle of odd symmetry.

Table I

x	0	(0, 1)	1	$(1, \sqrt{3})$	$\sqrt{3} \approx 1.73$	$(\sqrt{3}, 3)$	3	$(3, +\infty)$
y	0	−	$\pm\infty$	+	$\frac{\sqrt{3}}{\sqrt[3]{2}} \approx 1.37$	+	1.5	+
y'	−	−	non-exist	−	0	+	+	+
y''	0	−	non-exist	+	+	+	0	−
Conclusions	Point of inflection	Function decreases; graph is concave down	Discontinuity	Function decreases; graph is concave up	Min. point	Function increases; graph is concave up	Point of inflection	Function increases; graph is concave down

Using the results ot the investigation, we construct the graph of the function (Fig. 33).

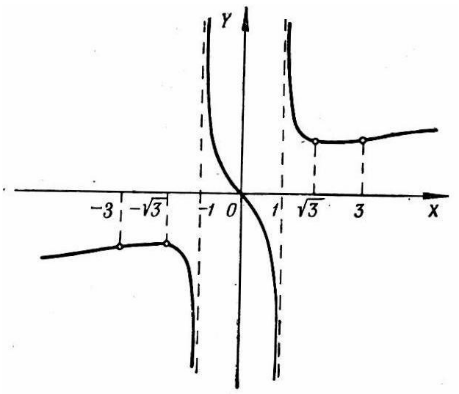

Fig. 33

Example2. Graph the function$y = \frac{\ln x}{x}$.

Solution:

a) The domain of definition of the function is 0 < x <∞.

b) There are no discontinuities in the domain of definition, but as we approach its boundary point (x=0) we have

$$\lim_{x\to 0} y = \lim_{x\to 0} \frac{\ln x}{x} = -\infty,$$

whence, the straight line *x* = 0 (the ordinate axis) is a vertical asymptote.

c)We seek the right asymptote (there is none on the left hand side, since *x* cannot tend to $-\infty$)

$$k = \lim_{x\to\infty} \frac{y}{x} = 0, \;\; b = \lim_{x\to\infty} y = 0.$$

The right asymptote is the abscissa *y* = 0*.*

d) We find the critical points and have$y' = \frac{1-\ln x}{x^2}$, $y'' = \frac{2\ln x-3}{x^3}$;y' and y" exist at all points of the domain of definition of the function and

$$y' = 0, when \ln x = 1, i.e. when x = e; y'' = 0, when \ln x = \frac{3}{2}, i.e. x = e^{\frac{3}{2}}.$$

We make a table, including the characteristic points (Table II).

Table 2

x	0	(0, 1)	1	(1, e)	$e \approx 2.72$	$\left(e, e^{\frac{3}{2}}\right)$	$e^{\frac{3}{2}} \approx 4.49$	$\left(e^{\frac{3}{2}}, +\infty\right)$
y	$-\infty$	$-$	0	$+$	$\frac{1}{e} \approx 0.37$	$+$	$\frac{3}{2\sqrt{e^3}} \approx 0.33$	$+$
y'	nonexist.	$+$	$+$	$+$	0	$-$	$-$	$-$
y''	nonexist.	$-$	$-$	$-$	$-$	$-$	0	$+$
Conclusions	Boundary point of domain of def. of function. Vertical asymptote	Funct. increases; graph is concave down	Funct. incr.; graph is concave down	Funct. incr.; graph is concave down	Max. point of funct.	Funct. decr.; graph is concave down	Point of inflection	Function decreases; graph is concave up

In addition to the characteristic points, it is useful to find the points of intersection of the curve with the co-ordinate axes. Setting y=0, we find x=1 (the poini of intersection of the curve with the abscissa); the curve does not intersect the ordinate.

e) Using these results, we construct the graph of the function (Fig. 34).

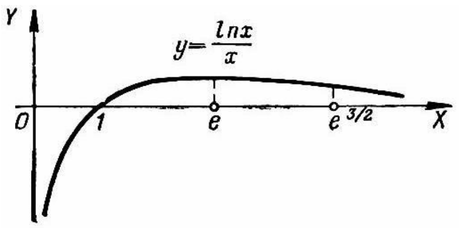

Fig. 34

Exercises 916 - 963

Graph the following functions and determine for each function its domain of definition, discontinuities, extreme points, intervals of increase and decrease, points of inflection of its graph, the direction of concavity, and also the asymptotes.

916. $y=x^3-3x^2$.

917. $y=\frac{6x^2-x^4}{9}$.

918. $y=(x-1)^2(x+2)$.

919. $y=\frac{(x-2)^2(x+4)}{4}$.

920. $y=\frac{(x^2-5)^3}{125}$.

921. $y=\frac{x^2-2x+2}{x-1}$.

922. $y=\frac{x^4-3}{x}$.

923. $y=\frac{x^4+3}{x}$.

924. $y=x^2+\frac{2}{x}$.

925. $y=\frac{1}{x^2+3}$.

926. $y=\frac{8}{x^2-4}$.

927. $y=\frac{4x}{4+x^2}$.

928. $y=\frac{4x-12}{(x-2)^2}$.

929. $y=\frac{x}{x^2-4}$.

930. $y=\frac{16}{x^2(x-4)}$.

931. $y=\frac{3x^4+1}{x^3}$.

932. $y=\sqrt{x}+\sqrt{4-x}$.

933. $y=\sqrt{8+x}-\sqrt{8-x}$.

934. $y=x\sqrt{x+3}$.

935. $y=\sqrt{x^3-3x}$.

936. $y=\sqrt[3]{1-x^2}$.

937. $y=\sqrt[3]{1-x^3}$.

938. $y=2x+2-3\sqrt[3]{(x+1)^2}$.

939. $y=\sqrt[3]{x+1}-\sqrt[3]{x-1}$.

940. $y=\sqrt[3]{(x+4)^2}-\sqrt[3]{(x-4)^2}$.

941. $y=\sqrt[3]{(x-2)^2}+\sqrt[3]{(x-4)^2}$.

942. $y=\frac{4}{\sqrt{4-x^2}}$.

943. $y=\frac{8}{x\sqrt{x^2-4}}$.

944. $y=\frac{x}{\sqrt[3]{x^2-1}}$.

945. $y=\frac{x}{\sqrt[3]{(x-2)^2}}$.

946. $y=xe^{-x}$.

947. $y=\left(a+\frac{x^2}{a}\right)e^{\frac{x}{a}}$.

948. $y=e^{8x-x^2-14}$.

949. $y=(2+x^2)e^{-x^2}$.

950. $y=2|x|-x^2$.

951. $y=\frac{\ln x}{\sqrt{x}}$.

952. $y=\frac{x^2}{2}\ln\frac{x}{a}$.

953. $y=\frac{x}{\ln x}$.

954. $y=(x+1)\ln^2(x+1)$.

955. $y=\ln(x^2-1)+\frac{1}{x^2-1}$.

956. $y=\ln\frac{\sqrt{x^2+1}-1}{x}$.

957. $y=\ln(1+e^{-x})$.

958. $y=\ln\left(e+\frac{1}{x}\right)$.

959. $y=\sin x+\cos x$.

960. $y=\sin x+\frac{\sin 2x}{2}$.

961. $y=\cos x-\cos^2 x$.

962. $y=\sin^3 x+\cos^3 x$.

963. $y=\frac{1}{\sin x+\cos x}$.

916. $y_{max}=0$ when $x=0$; $y_{min}=-4$ when $x=2$; point of inflection, $M_1(1, -2)$. 917. $y_{max}=1$ when $x=\pm\sqrt{3}$; $y_{min}=0$ when $x=0$; points of inflection $M_{1,2}\left(\pm 1, \frac{5}{9}\right)$. 918. $y_{max}=4$ when $x=-1$; $y_{min}=0$ when $x=1$; point of inflection, $M_1(0, 2)$. 919. $y_{max}=8$ when $x=-2$; $y_{min}=0$ when $x=2$; point of inflection, $M(0, 4)$. 920. $y_{min}=-1$ when $x=0$; points of inflection $M_{1,2}(\pm\sqrt{5}, 0)$ and $M_{3,4}\left(\pm 1, -\frac{64}{125}\right)$. 921. $y_{max}=-2$ when $x=0$; $y_{min}=2$ when $x=2$; asymptotes, $x=1$, $y=x-1$. 922. Points of inflection $M_{1,2}(\pm 1, \mp 2)$; asymptote $x=0$. 923. $y_{max}=-4$ when $x=-1$; $y_{min}=4$ when $x=1$; asymptote, $x=0$. 924. $y_{min}=3$ when $x=1$; point of inflection, $M(-\sqrt[3]{2}, 0)$; asymptote, $x=0$. 925. $y_{max}=\frac{1}{3}$ when $x=0$; points of inflection, $M_{1,2}\left(\pm 1, \frac{1}{4}\right)$; asymptote, $y=0$. 926. $y_{max}=-2$ when $x=0$; asymptotes, $x=\pm 2$ and $y=0$. 927. $y_{min}=-1$ when $x=-1$; $y_{max}=1$ when $x=1$; points of inflection, $O(0, 0)$ and $M_{1,2}\left(\pm 2\sqrt{3}, \pm\frac{\sqrt{3}}{2}\right)$; asymptote, $y=0$. 928. $y_{max}=1$ when $x=4$; point of inflection, $M\left(5, \frac{8}{9}\right)$; asymptotes, $x=2$ and $y=0$. 929. Point of inflection, $O(0, 0)$; asymptotes, $x=\pm 2$ and $y=0$. 930. $y_{max}=-\frac{27}{16}$ when $x=\frac{8}{3}$; asymptotes, $x=0$, $x=4$ and $y=0$. 931. $y_{max}=-4$ when $x=-1$; $y_{min}=4$ when $x=1$; asymptotes, $x=0$ and $y=3x$. 932. $A(0, 2)$ and $B(4, 2)$ are end-points; $y_{max}=2\sqrt{2}$ when $x=2$. 933. $A(-8, -4)$ and $B(8, 4)$ are end-points. Point of inflection, $O(0, 0)$. 934. End-point, $A(-3, 0)$; $y_{min}=-2$ when $x=-2$. 935. End-points, $A(-\sqrt{3}, 0)$, $O(0, 0)$ and $B(\sqrt{3}, 0)$; $y_{max}=\sqrt{2}$ when $x=-1$; point of inflection, $M\left(\sqrt{3+2\sqrt{3}}, \sqrt{6\sqrt{1+\frac{2}{\sqrt{3}}}}\right)$. 936. $y_{max}=1$ when $x=0$; points of inflection, $M_{1,2}(\pm 1, 0)$. 937. Points of inflection, $M_1(0, 1)$ and $M_2(1, 0)$; asymptote, $y=-x$. 938. $y_{max}=0$ when $x=-1$; $y_{min}=-1$ (when $x=0$). 939. $y_{max}=2$ when $x=0$; points of inflection, $M_{1,2}(\pm 1, \sqrt[3]{2})$; asymptote, $y=0$. 940. $y_{min}=-4$ when $x=-4$; $y_{max}=4$ when $x=4$; point of inflection, $O(0, 0)$; asymptote, $y=0$. 941. $y_{min}=\sqrt[3]{4}$ when $x=2$, $y_{min}=\sqrt[3]{4}$ when $x=4$; $y_{max}=2$ when $x=3$. 942. $y_{min}=2$ when $x=0$; asymptote, $x=\pm 2$. 943. Asymptotes, $x=\pm 2$ and $y=0$. 944. $y_{min}=\frac{\sqrt{3}}{\sqrt[3]{2}}$ when $x=\sqrt{3}$;

$y_{max}=-\frac{\sqrt{3}}{\sqrt[3]{2}}$ when $x=-3$; points of inflection, $M_1\left(-3, -\frac{3}{2}\right)$, $O(0, 0)$ and $M_2\left(3, \frac{3}{2}\right)$; asymptotes, $x=\pm1$. **945.** $y_{min}=\frac{3}{\sqrt[3]{2}}$ when $x=6$; point of inflection, $M\left(12, \frac{12}{\sqrt[3]{100}}\right)$; asymptote, $x=2$. **946.** $y_{max}=\frac{1}{e}$ when $x=1$; point of inflection, $M\left(2, \frac{2}{e^2}\right)$; asymptote, $y=0$. **947.** Points of inflection, $M_1\left(-3a, \frac{10a}{e^3}\right)$ and $M_2\left(-a, \frac{2a}{e}\right)$; asymptote, $y=0$. **948.** $y_{max}=e^2$ when $x=4$; points of inflection, $M_{1,2}\left(\frac{8\pm2\sqrt{2}}{2}, e^{\frac{3}{2}}\right)$; asymptote, $y=0$. **949.** $y_{max}=2$ when $x=0$; points of inflection, $M_{1,2}\left(\pm1, \frac{3}{e}\right)$. **950.** $y_{max}=1$ when $x=\pm1$; $y_{min}=0$ when $x=0$. **951.** $y_{max}=0,74$ when $x=e^2\approx7.39$; point of inflection, $M(e^{8/3}\approx14.39, 0.70)$; asymptotes, $x=0$ and $y=0$. **952.** $y_{min}=-\frac{a^2}{4e}$ when $x=\frac{a}{\sqrt{e}}$; point of inflection, $M\left(\frac{a}{\sqrt{e^3}}, -\frac{3a^2}{4e^2}\right)$. **953.** $y_{min}=e$ when $x=e$; point of inflection, $M\left(e^2, \frac{e^2}{2}\right)$; asymptote, $x=1$; $y\to0$ when $x\to0$. **954.** $y_{max}=\frac{4}{e^2}\approx0.54$ when $x=\frac{1}{e^2}-1\approx-0.86$; $y_{min}=0$ when $x=0$; point of inflection, $M\left(\frac{1}{e}-1\approx-0.63; \frac{1}{e}\approx0.37\right)$; $y\to0$ as $x\to-1+0$ (limiting end-point). **955.** $y_{min}=1$ when $x=\pm\sqrt{2}$; points of inflection, $M_{1,2}(\pm1.89, 1.33)$; asymptotes, $x=\pm1$. **956.** Asymptote, $y=0$. **957.** Asymptotes, $y=0$ (when $x\to+\infty$) and $y=-x$ (as $x\to-\infty$). **958.** Asymptotes, $x=-\frac{1}{e}$, $x=0$, $y=1$; the function is not defined on the interval $\left[-\frac{1}{e}, 0\right]$. **959.** Periodic function with period 2π. $y_{min}=-\sqrt{2}$ when $x=\frac{5}{4}\pi+2k\pi$; $y_{max}=\sqrt{2}$ when $x=\frac{\pi}{4}+2k\pi$ $(k=0, \pm1, \pm2, \ldots)$; points of inflection, $M_k\left(\frac{3}{4}\pi+k\pi, 0\right)$. **960.** Periodic function with period 2π. $y_{min}=-\frac{3}{4}\sqrt{3}$ when $x=\frac{5}{3}\pi+2k\pi$; $y_{max}=\frac{3}{4}\sqrt{3}$ when $x=\frac{\pi}{3}+2k\pi$ $(k=0, \pm1, \pm2, \ldots)$; points of inflection, $M_k(k\pi, 0)$ and $N_k\left(\arccos\left(-\frac{1}{4}\right)+2k\pi, \frac{3}{16}\sqrt{15}\right)$. **961.** Periodic function with period 2π. On the interval $[-\pi, \pi]$, $y_{max}=\frac{1}{4}$ when $x=\pm\frac{\pi}{3}$; $y_{min}=-2$ when $x=\pm\pi$; $y_{min}=0$ when $x=0$; points of inflection, $M_{1,2}(\pm0.57, 0.13)$ and $M_{3,4}(\pm2.20, -0.95)$. **962.** Odd periodic function with period 2π. On interval $[0, 2\pi]$, $y_{max}=1$ when $x=0$; $y_{min}=0.71$, when $x=\frac{\pi}{4}$; $y_{max}=1$ when

$x=\frac{\pi}{2}$; $y_{min}=-1$ when $x=\pi$; $y_{max}=-0.71$ when $x=\frac{5}{4}\pi$; $y_{min}=-1$ when $x=\frac{3}{2}\pi$; $y_{max}=1$ when $x=2\pi$; points of inflection, $M_1(0.36,\ 0.86)$; $M_2(1.21,\ 0.86)$; $M_3(2.36,\ 0)$; $M_4(3.51,\ -0.86)$; $M_5(4.35,\ -0.86)$; $M_6(5.50,\ 0)$. **963.** Periodic function with period 2π. $y_{min}=\frac{\sqrt{2}}{2}$ when $x=\frac{\pi}{4}+2k\pi$; $y_{max}=-\frac{\sqrt{2}}{2}$ when $x=-\frac{3}{4}\pi+2k\pi$ $(k=0,\ \pm 1,\ \pm 2,\ \ldots)$; asymptotes, $x=\frac{3}{4}\pi+k\pi$.