First Order Ordinary Differential Equation

Introduction

Prerequisites: You will need to know about trigonometry, differentiation, integration, complex numbers in order to make the most of this teach-yourself resource.

We are looking at equations involving a function $y(x)$ and its first derivative:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1}$$

We want to find $y(x)$, either explicitly if possible, or otherwise implicitly.

Direct Integration

This method is used to solve ODEs in the form:

$$\frac{dy}{dx} = f(x)$$

These ODEs can be solved as follows:

$$\begin{aligned} dy &= f(x)dx \\ \text{so: } y &= \int f(x)dx \end{aligned}$$

Example:

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 - 6x + 5 \\ y &= x^3 - 3x^2 + 5x + C \text{ with } C \text{ constant of integration} \end{aligned}$$

The constant of integration can be anything, unless you have boundary conditions. In the previous example, if you have y(0)=0 then:

$$y(0) = C = 0$$

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Separation of Variables

This method is used to solve ODEs in the form:

$$\frac{dy}{dx} = f(x)g(y)$$

These ODEs can be solved as follows:

$$\begin{aligned} \frac{dy}{g(y)} &= f(x)dx \\ \int \frac{dy}{g(y)} &= \int f(x)dx \end{aligned}$$

Example:

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x}{y+1} \\ (y+1)dy &= 2xdx \\ \int (y+1)dy &= \int 2xdx \\ \frac{1}{2}y^2 + y &= x^2 + C \text{ with C constant of integration} \end{aligned}$$

Homogeneous Equations

A first order homogeneous differential equation is a differential equation in the form:

$$\frac{dy}{dx} = f(\frac{y}{x})$$

These ODEs can be solved by making the substitution $y(x) = v(x) \cdot x$ where v is a function of x. Then we have:

Using the product rule: $\frac{dy}{dx} = \frac{dv}{dx} \cdot x + v$

Inserting in the ODE: $\frac{dv}{dx} \cdot x + v = f(v)$

Re-arranging: $\frac{dv}{f(v) - v} = \frac{dx}{x}$

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$$

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Example

$$\frac{dy}{dx} = \frac{x+3y}{2x}$$

Rearranging, we get: $\frac{dy}{dx} = \frac{1}{2} + \frac{3}{2}\frac{y}{x}$

$$x\frac{dv}{dx} + v = \frac{1}{2} + \frac{3}{2}v$$

$$x\frac{dv}{dx} = \frac{1}{2} + \frac{1}{2}v$$

$$\frac{dv}{1+v} = \frac{dx}{2x}$$

$$\ln(1+v) = \frac{1}{2}\ln(x) + C = \ln(x^{1/2}) + C \text{ with} C \text{ constant of integration}$$

Taking the log on both sides: $v = e^C x^{1/2} - 1$

with $y = v \cdot x$, $y = kx^{3/2} - x$, with $k = e^C$

Integrating Factor

This method is used to solve ODEs in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

These ODEs can be solved as follows: multiply both sides of the equation by **the integrating factor:** $e^{\int P(x)dx}$. Then:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}$$

Now we see that, using the product rule and the chain rule:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = \frac{d}{dx}\left(e^{\int P(x)dx}y\right)$$

Therefore:

$$\frac{d}{dx}\left(e^{\int P(x)dx}y\right) = Q(x)e^{\int P(x)dx}$$

$$y = e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx$$

Example

$$\frac{dy}{dx} - y = x \text{ with } P(x) = -1 \text{ and } Q(x) = x$$

The integrating factor is $e^{\int -dx} = e^{-x}$ and:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{-x}$$

$$\frac{d}{dx}\left(ye^{-x}\right) = xe^{-x}$$

Using integration by part: $ye^{-x} = -e^{-x}(1+x) + C$ with C constant of integration

$$y = -(1+x) + Ce^x$$

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Bernouilli Equations

Bernouilli equations are of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

These equations are solved with the following method:

- Multiply both sides of the equation by y^{-n}, the equation becomes:

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

- Make the change of variable $v = y^{1-n}$, then:

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\text{so} \quad \frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$$

The latest form of the equation can be solved for v using the method of the integrating factor. Finally, y can be found from the relationship $y = v^{n-1}$.

Example:

$$\begin{aligned}
\frac{dy}{dx} + \frac{1}{x}y &= xy^2 \\
y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} &= x \\
v = y^{-1} \quad , & \quad \frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \\
\frac{dv}{dx} - \frac{v}{x} &= -x \\
\text{The integrating factor is IF} &= e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x} \\
\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2}v &= -1 \\
\frac{1}{x}v &= \int -dx \\
v &= -x^2 + Cx \\
y &= \frac{1}{-x^2 + Cx}
\end{aligned}$$

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Exercises

(a) $x\dfrac{dy}{dx} = 5x^3 + 4$

(b) $x(y-3)\dfrac{dy}{dx} = 4y$

(c) $(2y-x)\dfrac{dy}{dx} = 2x + y$ with $y(2) = 3$

(d) $x\dfrac{dy}{dx} - y = x^3 + 3x^2 - 2x$

(e) $(1+x^2)\dfrac{dy}{dx} + 3xy = 5x$ with $y(1) = 2$

(f) $2\dfrac{dy}{dx} + y = y^3(x-1)$

Answers

(a) $y = \dfrac{5}{3}x^3 + 4\ln x + C$

(b) $y - 3\ln y = 4\ln x + C$

(c) $(\dfrac{y}{x})^2 - \dfrac{y}{x} - 1 + x^{-2} = 0$

(d) $y = \dfrac{1}{2}x^3 + 3x^2 - 2x\ln x + Cx$

(e) $y = \dfrac{5}{3} + \dfrac{\sqrt{8}}{3}(1+x^2)^{-3/2}$

(f) $y = (x + Ce^{-x})^{-1/2}$

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