Introduction to the derivatives

1 Increment of the argument and increment of a function.

If x and x_1 are the values of the argument x and $y = f(x)$ and $y_1 = f(x_1)$ are the corresponding values of the function $y = f(x)$, then

$$\Delta x = x_1 - x$$

is called the increment of the argument x in the interval (x, x_1) and

$$\Delta y = y_1 - y \quad or \quad \Delta y = f(x_1) - f(x) = f(x + \Delta x) - f(x) \qquad (1)$$

is called the increment of the function y in the same interval (x, x_1) (Fig, 1, where $\Delta x = MA$ and $\Delta y = AN$).

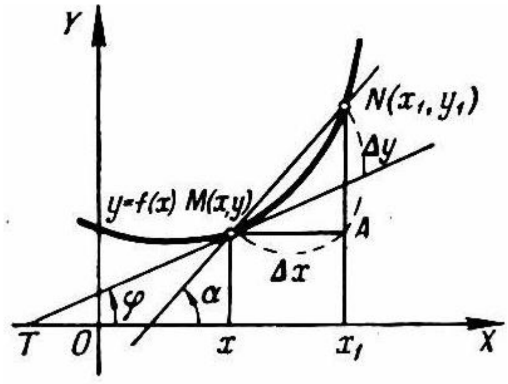

Fig. 1

The ratio $\frac{\Delta y}{\Delta x} = \tan\alpha$ is the slope of the secant MN of the graph of the function $y = f(x)$ and is called the mean rate of change of the function y over the interval $(x, x + \Delta x)$.

Example. Calculate for the function $y = x^2 - 5x + 6$ values Δx and Δy corresponding to a change in the argument: a) from $x = 1$ to $x = 1.1$, b) from $x = 3$ to $x = 2$.

Solution: We have

$a)\ \Delta x = 1{,}1 - 1 = 0{,}1;\quad \Delta y = (1{,}1^2 - 5\cdot 1{,}1 + 6) - (1^2 - 5\cdot 1 + 6) = -0{,}29;$

$b)\ \Delta x = 2 - 3 = -1;\quad \Delta y = (2^2 - 5\cdot 2 + 6) - (3^2 - 5\cdot 3 + 6) = 0.$

Example. Find for the hyperbola $y = 1/x$ the slope of the secant passing through the points $M(3, 1/3)$ and N (10, 1/10).

Solution: Here,

$$\Delta x = 10 - 3 = 7, \Delta y = \frac{1}{10} - \frac{1}{3} = -\frac{7}{30}, whence\ k = \frac{\Delta y}{\Delta x} = -\frac{1}{30}.$$

The derivative of the function $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and basic rules for derivatives

The derivative y' of a function $y = f(x)$ with respect to the argument x is the limit of the ratio $\Delta y/\Delta x$ when Δx approaches zero.

Notation: $y',\quad y'(x),\quad f'(x),\quad y'_x,\quad f'_x,\quad \frac{dy}{dx},\quad \frac{d}{dx}(y)$

Table of derivatives of basic functions:

1. $(c)' = 0$	9. $(\tan x)' = \frac{1}{cos^2 x}$
2. $(x^n)' = nx^{n-1}$	10. $(cotx)' = -\frac{1}{sin^2 x}$
3. $(e^x)' = e^x$	11. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
4. $(a^x)' = a^x \ln a$	12. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
5. $(\ln x)' = \frac{1}{x}$	13. $(\arctan x)' = \frac{1}{1+x^2}$
6. $(log_a x)' = \frac{1}{x \ln a}$	14. $(\text{arccot}\, x)' = -\frac{1}{1+x^2}$
7. $(\sin x)' = \cos x$	
8. $(\cos x)' = -\sin x$	

1.1 The Constant Rule

Example. Find the derivative of $f(x) = 8, g(x) = \pi$.

Solution: This is just a one-step application of the rule: $f'(x) = 8' = 0, \quad g'(x) = \pi' = 0.$

1.2 The Power Rule

Example. Find the derivative of $f(x) = x^4$

Solution: Use the power rule with $n = 4$, so $f'(x) = (x^4)' = 4x^{4-1} = 4x^3$

Example. Find the derivative of $f(x) = \frac{1}{x^2}$

Solution: To use the power rule the formula of $f(x)$ has to be rearrenged as $f(x) = x^{-2}$ and now we can use the power rule with $n = -2$

$$f'(x) = (x^{-2})' = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

Example. Find the derivative of $f(x) = \sqrt{x}$

Solution: To use the power rule the formula of $f(x)$ has to be rearrenged as $f(x) = x^{1/2}$ and now we can use the power rule with $n = 1/2$

$$f'(x) = \left(x^{1/2}\right)' = 1/2 \cdot x^{\frac{1}{2}-1} = 1/2 \cdot x^{-1/2} = \frac{1}{2\sqrt{x}}$$

2 The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant.

Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x));$$

that is,

$$[\ f(x)+g(x)\ ]' = f'(x)+g'(x).$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g :

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x));$$

that is,

$$[\ f(x)-g(x)\ ]' = f'(x)-g'(x).$$

Constant Multiple Rule. The derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x));$$

that is,

$$[\ kf(x)\ ]' = kf'(x).$$

Example. Find the derivative of $f(x) = 2x^5 + 7$

Solution: We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers

$f'^{(x)} = (2x^5 + 7)' = (2x^5)' + (7)'$ Apply the sum rule;

$= 2(x^5)' + (7)'$ Apply the constant multiple rule;

$= 2(5x^4) + 0$ Apply the power rule and the constant rule;

$= 10x^4$ Apply the constant multiple rule.

Example. Find the derivative of $f(x) = 2x^3 - 6x^2 + 3$

Solution: Use the preceding example as a guide $f'(x) = 6x^2 - 12x$

2.1 Derivatives of Exponential and Logarithmic Functions

Example. Find the derivative of $f(x) = 2^x$

Solution: $f'(x) = (2^x)' = 2^x \cdot \ln 2$

Example. Find the derivative of $f(x) = \log_2 x$

Solution: $f'(x) = (\log_2 x)' = \frac{1}{x \cdot \ln 2}$

2.2 Derivatives of Trigonometric Functions

Example. Find the derivative of $f(x) = 2\sin x - 7\cos x$

Solution: $f'(x) = (2\sin x - 7\cos x)' = 2\cos x + 7\sin x$

Example. Find the derivative of $f(x) = \tan x - \cot x$

Solution: $f'(x) = (\tan x - \cot x)' = \frac{1}{(\cos x)^2} + \frac{1}{(\sin x)^2} = \frac{1}{(\sin x)^2 \cdot (\cos x)^2}$

3 The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and difference rules, the product rule does not follow this pattern

Let f(x) and g(x) be differentiable function.

Product Rule

If the two functions $f(x)$ and $g(x)$ are differentiable (*i.e.* the derivative exist) then the product is differentiable and,

$$(fg)' = f'g + fg'$$

Example Applying the Product Rule to Constant Functions

For $j(x) = f(x)g(x)$, use the product rule to find $j'(2)$ if $f(2) = 3$, $f'(2) = -4$, $g(2) = 1$, and $g'(2) = 6$.

Solution

Since $j(x) = f(x)g(x)$, $j'(x) = f'(x)g(x) + g'(x)f(x)$, and hence

$$j'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14. \tag{3.3.40}$$

Example Applying the Product Rule to Binomials

For $j(x) = (x^2 + 2)(3x^3 - 5x)$, find $j'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

Solution

If we set $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$, then $f'(x) = 2x$ and $g'(x) = 9x^2 - 5$. Thus,

$j'(x) = f'(x)g(x) + g'(x)f(x) = (2x)(3x^3 - 5x) + (9x^2 - 5)(x^2 + 2)$.

Simplifying, we have

$$j'(x) = 15x^4 + 3x^2 - 10. \tag{3.3.41}$$

To check, we see that $j(x) = 3x^5 + x^3 - 10x$ and, consequently, $j'(x) = 15x^4 + 3x^2 - 10$.

4 The Quotient Rule

Let f(x) and g(x) be differentiable function.

Quotient Rule

If the two functions $f(x)$ and $g(x)$ are differentiable (*i.e.* the derivative exist) then the quotient is differentiable and,

$$\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$$

Example Differentiate each of the following functions.

(a) $W(z) = \dfrac{3z + 9}{2 - z}$

(b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

(c) $f(x) = \dfrac{4}{x^6}$

(d) $y = \dfrac{w^6}{5}$

(a) $W(z) = \dfrac{3z + 9}{2 - z}$

There isn't a lot to do here other than to use the quotient rule. Here is the work for this function.

$$\begin{aligned} W'(z) &= \frac{3(2 - z) - (3z + 9)(-1)}{(2 - z)^2} \\ &= \frac{15}{(2 - z)^2} \end{aligned}$$

(b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

Again, not much to do here other than use the quotient rule. Don't forget to convert the square root into a fractional exponent.

$$\begin{aligned} h'(x) &= \frac{4\left(\frac{1}{2}\right)x^{-\frac{1}{2}}\left(x^2 - 2\right) - 4x^{\frac{1}{2}}(2x)}{\left(x^2 - 2\right)^2} \\ &= \frac{2x^{\frac{3}{2}} - 4x^{-\frac{1}{2}} - 8x^{\frac{3}{2}}}{\left(x^2 - 2\right)^2} \\ &= \frac{-6x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}}{\left(x^2 - 2\right)^2} \end{aligned}$$

(c) $f(x) = \frac{4}{x^6}$

It seems strange to have this one here rather than being the first part of this example given that it definitely appears to be easier than any of the previous two. In fact, it is easier. There is a point to doing it here rather than first. In this case there are two ways to do compute this derivative. There is an easy way and a hard way and in this case the hard way is the quotient rule. That's the point of this example.

Let's do the quotient rule and see what we get.

$$f'(x) = \frac{(0)(x^6) - 4(6x^5)}{(x^6)^2} = \frac{-24x^5}{x^{12}} = -\frac{24}{x^7}$$

Now, that was the "hard" way. So, what was so hard about it? Well actually it wasn't that hard, there is just an easier way to do it that's all. However, having said that, a common mistake here is to do the derivative of the numerator (a constant) incorrectly. For some reason many people will give the derivative of the numerator in these kinds of problems as a 1 instead of 0! Also, there is some simplification that needs to be done in these kinds of problems if you do the quotient rule.

The easy way is to do what we did in the previous section.

$$f(x) = 4x^{-6} \qquad f'(x) = -24x^{-7} = -\frac{24}{x^7}$$

Either way will work, but I'd rather take the easier route if I had the choice.

(d) $y = \frac{w^6}{5}$

This problem also seems a little out of place. However, it is here again to make a point. Do not confuse this with a quotient rule problem. While you can do the quotient rule on this function there is no reason to use the quotient rule on this. Simply rewrite the function as

$$y = \frac{1}{5}w^6$$

and differentiate as always.

$$y' = \frac{6}{5}w^5$$

5 Exercises.

Exercises 1-5.

1. Find the increment of the function $y = x^2$ which corresponds to a given change in argument:

a) from $x = 1$ to $x = 2$; b) from $x = 1$ to $x = 1{,}1$; c) from $x = 1$ to $x = 1 + h$.

2. Find Δy of the function $y = \sqrt[3]{x}$ if

a) $x = 0,\ \Delta x = 0{,}001;\quad b)\ x = 8,\ \Delta x = -9;\quad c)\ \ x = a,\ \Delta x = h.$

3. Why can we determine for the function $y = 2x + 3$ the increment Δy, if all we know is the corresponding increment $\Delta x = 5$, while for the function $y = x^2$ this cannot be done?

4. Find the increment Δy and the ratio $\frac{\Delta y}{\Delta x}$ for the functions:

$$a)\ y = \frac{1}{(x^2-2)^2}\ \ for\ x = 1\ and\ \Delta x = 0{,}4; b)\ y = \sqrt{x}\ for\ x = 0\ and\ \Delta x = 0{,}0001.$$

5. Find Δy and $\Delta y/\Delta x$ which correspond to a change in argument from x to Δx for the functions:

$$a)\ y = ax + b;\quad b)\ y = x^3;\quad c)\ y = \frac{1}{x^2};\quad d)\ y = \sqrt{x};\quad e)\ y = 2^x;\quad f)\ y = \ln x.$$

Answers 1-5.

a) 3; **b)** 0,21; **c)** $2h + h^2$. **2. a)** 0,1 **b)** -3; **c)** $\sqrt[3]{a+h} - \sqrt[3]{a}$. **4. a)** 624; 1560; **b)** 0,01; 100; **c)** -1;

0,000011. **5.** a) $a\Delta x$; b) $3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$; $3x^2 + 3x\Delta x + (\Delta x)^2$; c) $-\frac{2x\Delta x+(\Delta x)^2}{x^2(x+\Delta x)^2}$; $-\frac{2x+\Delta x}{x^2(x+\Delta x)^2}$; d) $\sqrt{x + \Delta x} - \sqrt{x}$; $\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}$; e) $2^x(2^{(\Delta x)} - 1)$; $\frac{2^x(2^{(\Delta x)}-1)}{\Delta x}$; f) $\ln\frac{x+\Delta x}{x}$; $\frac{1}{\Delta x}\ln\frac{x+\Delta x}{x}$.

Find the derivatives of the following functions using tabel of derivatives and properties.

Exercises 6-19 (Algebraic functions)

6. $y = x^5 - 4x^3 + 2x - 3$. **7.** $y = \frac{1}{4} - \frac{1}{3}x + x^2 - 0{,}5x^4$. **8.** $y = ax^2 + bx + c$. **9.** $y = -\frac{5x^3}{a}$.

10. $y = at^m + bt^{n+m}$. **11.** $y = \frac{ax^2+b}{\sqrt{a^2+b^2}}$. **12.** $y = \frac{\pi}{x} + \ln 2$. **13.** $y = 3x^{\frac{2}{3}} - 2x^{\frac{5}{2}} + x^{-3}$. **14.** $y = x^2\sqrt[3]{x^2}$.

15. $y = \frac{a}{\sqrt[3]{x^2}} - \frac{b}{x\sqrt[3]{x}}$. **16.** $y = \frac{a+bx}{c+dx}$. **17.** $y = \frac{2x+3}{x^2-5x+5}$. **18.** $y = \frac{2}{2x-1} - \frac{1}{x}$. **19.** $y = \frac{1+\sqrt{z}}{1-\sqrt{z}}$.

Exercises 20-27 (Inverse Circular and Trigonometric Functions)

20. $y = 5\sin x + 3\cos x$. **21.** $y = \tan x - \cot x$. **22.** $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$. **23.** $y = 2t\sin t - (t^2 - 2)\cos t$.

24. $y = \arctan x + \operatorname{arccot} x$. **25.** $y = x \cdot \cot x$. **26.** $y = x \cdot \arcsin x$. **27.** $y = \dfrac{(1 + x^2) \cdot \arctan x - x}{2}$

Answers 20-27. **20.** $y' = 5\cos x - 3\sin x$. **21.** $y' = \frac{2}{(\sin 2x)^2}$. **22.** $y' = \frac{-2}{(\sin x-\cos x)^2}$. **23.** $y' = t^2 \sin t$.

24. $y' = 0$. **25.** $y' = \cot x - \dfrac{x}{\sin^2 x}$. **26.** $y' = \arcsin x + \dfrac{x}{\sqrt{1-x^2}}$. **27.** $y = x \cdot \arctan x$

Exercises 28- 38 (Exponential and Logarithmic Functions)

28. $y = x^7 \cdot e^x$. **29.** $y = (x - 1)e^x$. **30.** $y = \dfrac{e^x}{x^2}$. **31.** $y = \dfrac{x^5}{e^x}$. **32.** $y = e^x \cdot \cos x$. **33.** $y = (x^2 - 2x + 2)e^x$.

34. $y = e^x \cdot \arcsin x$. **35.** $y = \dfrac{x^2}{\ln x}$. **36.** $y = x^3 \cdot \ln x - \dfrac{x^3}{3}$ **37.** $y = \dfrac{1}{x} + 2\ln x - \dfrac{\ln x}{x}$.

38. $y = \ln x \cdot \log x - \ln a \cdot \log_a x$.

Answers 28-38.

28. $y = x^6 \cdot e^x(x + 7)$. **29.** $x \cdot e^x$. **30.** $e^x \cdot \dfrac{x-2}{x^3}$. **31.** $\dfrac{5x^4 - x^5}{e^x}$. **32.** $e^x(\cos x - \sin x)$.

33. $x^2 \cdot e^x$. **34.** $e^x\left(\arcsin x + \dfrac{1}{\sqrt{1-x^2}}\right)$. **35.** $\dfrac{x(2\ln x - 1)}{\ln^2 x}$. **36.** $3x^2 \cdot \ln x$. **37.** $\dfrac{2}{x} + \dfrac{\ln x}{x^2} - \dfrac{2}{x^2}$.

38. $\dfrac{2\ln x}{x \cdot \ln 10} - \dfrac{1}{x}$

6 At the point

$$y'(x_0),\ \text{või}\ \ y'(x)\big|_{x=x_0},\ \ \text{või}\ \ f'(x_0),\ \ \text{või}\ \ f'(x)\big|_{x=x_0}$$