1. Quick review of composite functions

1.1. Composite Function

A function is *composite* if you can write it as $f(g(x))$.

In other words, it is a function within a function, or a function of a function. For example, $\cos(x^2)$ is composite, because if we let $f(x) = \cos(x)$ and $g(x) = x^2$, then $\cos(x^2) = f(g(x))$.

$$\underbrace{\cos(\overbrace{x^2}^{\text{inner}})}_{\text{outer}}$$

Function g is the function within f, so we call g the „inner“ function and f the „outer“ fuction.

Common mistake: Not recognizing whether a function is composite or not

Usually, the only way to differentiate a composite function is using the chain rule. If we don't recognize that a function is composite and that the chain rule must be applied, we will not be able to differentiate correctly.

On the other hand, applying the chain rule on a function that isn't composite will also result in a wrong derivative.

Especially with transcendental functions (e.g., trigonometric and logarithmic functions), students often confuse compositions like $\ln(\sin(x))$ products like $\ln(x) \cdot \sin(x)$.

On the other hand, $\cos(x) \cdot x^2$ is *not* a composite function. It is the *product* of $f(x) = \cos(x)$ and $g(x) = x^2$, but neither of the functions is within the other one.

Example. Have a look at the function $f(x) = (x^2 + 1)^{17}$. We can think of this function as being the result of combining two functions. Another way of representing this would be with a diagram like

$$x \overset{g}{\longmapsto} x^2 + 1 \overset{h}{\longmapsto} (x^2 + 1)^{17}.$$

We start off with x. Te fuction g takes x to $x^2 + 1$, and the function h then takes $x^2 + 1$ to $(x^2 + 1)^{17}$.

Exercises. Work out $f(g(x))$ and $g(f(x))$ for the following pairs of functions:

1. $f(x) = 3x,\ g(x) = 2x^2 + 1$
2. $f(x) = e^{4x},\ g(x) = \sqrt{x}$
3. $f(x) = \sin x\,,\ g(x) = 1/x$

1.2. Order of composition

The order in which we compose functions makes a big difference to the result. You can see this if we change the order of the functions in the first example. We have taken $f(x) = x^2$ and $g(x) = x + 3$. Then

$$f(g(x)) = f(x+3) = (x+3)^2 = x^2 + 6x + 9$$

and

$$g(f(x)) = g(x^2) = x^2 + 3.$$

In general $g(f(x))$ is not equal to $f(g(x))$.

Exercises. Work out $g(f(x))$ for the following pairs of functions and compare the results to those you obtained for Exercises 1-3.

4. $f(x) = 3x, \quad g(x) = 2x^2 + 1$
5. $f(x) = e^{4x}, \quad g(x) = \sqrt{x}$
6. $f(x) = \sin x, \quad g(x) = 1/x$